PROJECTIVE DIFFERENTIAL GEOMETRY OF CURVED SURFACES*

(FIFTH MEMOIR)

BY

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Introduction.

On December 13, 1880, DARBOUX presented to the French Academy of Sciences a note on the contact between curves and surfaces, which contains some very important results.† One of these may be stated as follows: if we consider all of the sections of a surface made by planes which pass through the same tangent, the locus of the conics, which osculate these sections at their common point of contact with the tangent, is a quadric having contact of the second order with the given surface. Two of these osculating conics moreover hyperosculate the plane section to which they belong, i. e., have fifth order contact with it. Finally, there exist (in general) twenty-seven plane sections through a point of a given surface whose osculating conics have sixth order contact with them. DARBOUX also discussed an interesting problem of contact between a quadric and a given surface, and determined a STEINER surface which has fourth order contact with the given surface at a given point. He further proved that the quadric surfaces, the ruled surfaces of the third order and STEINER's Roman surface of the fourth order are the only ones through every point of which there passes an infinity of conics lying entirely on the surface. The other results in DARBOUX's paper are of a metrical character and do not concern us here.

Two weeks later MOUTARD‡ announced the fact that the same questions treated by DARBOUX had been discussed by him in two oral communications made to the *Société philomathique* about 1865. The second of these was concerned with metrical geometry and calls for no discussion in this place. Moreover no trace of it exists in print. The first however contained precisely the theorems mentioned above concerning the conics which have contact of the fourth, fifth and sixth order with a given surface at a given point,

* Presented to the Society (Chicago), January 1, 1909.

† Comptes Rendus, vol. 91 (1880), p. 969. The detailed discussion of these results is to be found in the Bulletin des Sciences Mathematiques et Astronomiques, 2d series, vol. 4 (1880), p. 348.

‡ Comptes Rendus, vol. 91 (1880), p. 1055.

together with some additional more special remarks which follow from an application of these results to surfaces of the third order. These theorems were communicated by MOUTARD to PONCELET in a letter, a part of which was reproduced by the latter in the second volume of his *Applications d'analyse et de géometrie.* A part of this again was used by CHASLES in 1870 in his *Rapport sur le progrès de la géometrie.* These important theorems discovered by MOUTARD in 1863, it seems, were practically lost until rediscovered independently by DARBOUX in 1880. A very careful search on my part has failed to show me any trace of these theorems in the standard treatises on the theory of surfaces, which have appeared since 1880, or in the "*Encyklopädie.*" Nor have I seen any mention of these results in the general literature of the subject. Consequently I thought them new when I rediscovered them in the spring of 1908, and announced them as such at the summer-meeting of the SOCIETY in Urbana. I shall henceforth speak of the first of the above mentioned results as the theorem of MOUTARD.

The method of proof which I have adopted is closely allied to that of DARBOUX. But there are several essential points of difference. The tetrahedron of reference chosen by me is one whose geometrical significance has been fully elucidated by my previous work, while that of DARBOUX, which is not the same, was introduced by him in a purely analytic fashion. Since, moreover, these theorems are necessary parts of any general treatment of the projective differential geometry of a curved surface, such as has been presented for the first time in these five memoirs, since other results closely related to them cannot be presented except in this connection, and since the modifications which DARBOUX's theorems undergo in certain important cases are also included in this treatment it becomes necessary, even at the expense of some repetition, to publish these new proofs. In this way the concepts of the MOUTARD-DARBOUX theory can be brought into organic connection with the theory developed by myself in the first four memoirs. Another method of establishing this connection would consist in obtaining the geometric characteristics of the DARBOUX tetrahedron of reference. This will be left for another paper, as will also the detailed discussion of the exceptional case when the point of the surface considered is parabolic. The case in which all points of the surface are parabolic, i. e., in which the surface is developable, shall however be included in this paper at least as far as the principal features of the theory are concerned. This enables us to apply our theorems in detail to all analytic surfaces, including developables, only special points of certain surfaces being excluded.

§ 1. *Darboux's tangents of quadric osculation.*

Let P be a point of a surface S, and let R be a non-vanishing portion of S, finite or infinite in extent, such that P is one of its interior points, and such

that the homogeneous coördinates $x_1, \cdots, x_4$ of any point of R may be expressed as convergent series of positive integral powers of two parameters u and v. Moreover let the two asymptotic tangents of the surface at P be distinct. We shall then say that P is a *regular* point of the surface.

If we introduce non-homogeneous coördinates

$$x = \frac{x_1}{x_4}, \qquad y = \frac{x_2}{x_4}, \qquad z = \frac{x_3}{x_4},$$

these may be chosen in such a way that one of them, say z, assumes the form of a series proceeding according to positive integral powers of the other two, x and y. If, moreover, neither of the asymptotic tangents of P has more than three consecutive points in common with the surface at P, this series may by a projective transformation be put into the form

$$z = xy + \tfrac{1}{6}(x^3 + y^3) + \tfrac{1}{24}(Ix^4 + Jy^4) + \cdots, \tag{1}$$

where I, J, and all further coefficients are absolute invariants of the surface. Any non-ruled surface will admit of such a development in the vicinity of one of its regular points. DARBOUX † mentions this development but makes use of a different one, viz.:

$$z = xy + x^3 + y^3 + xy(ax^2 + by^2) + \cdots, \tag{1a}$$

in the investigations mentioned in the introduction. The form (1) of the development is also derived by TRESSE ‡ but implicitly is already contained in a few remarks of HALPHEN's in his thesis on differential invariants.

The geometrical significance of the development (1), i. e., the geometrical definition of the tetrahedron of reference which gives rise to this development is explained in one of my previous papers.

Obviously the most general quadric involving three arbitrary constants α, β, γ, which has contact of the second order with S at P is given by an equation of the form

$$z = xy + z(\alpha x + \beta y + \gamma z), \tag{2}$$

which gives the development for z

$$z = xy + xy(\alpha x + \beta y) + xy(\alpha x + \beta y)^2 + \gamma x^2 y^2 + \cdots. \tag{3}$$

The projection of the curve of intersection of (1) and (2) upon the xy plane will be

$$\tfrac{1}{6}(x^3 + y^3) - xy(\alpha x + \beta y) + \tfrac{1}{24}(Ix^4 + Jy^4) - xy[(\alpha x + \beta y)^2 + \gamma xy] + \cdots = 0.$$

This curve has a triple point at P for all values of α and β. The three tangents of this triple point will coincide if and only if

* Second memoir. These Transactions, vol. 9 (1908), p. 103.

† DARBOUX, l. c., p. 353.

‡ A. TRESSE. *Sur les invariants différentiels des groupes continus de transformations*, Paris, 1893.

$\alpha = -\frac{1}{2}, \quad \beta = -\frac{1}{2}; \quad \alpha = -\frac{1}{2}\theta, \quad \beta = -\frac{1}{2}\theta; \quad \alpha = -\frac{1}{2}\theta^2, \quad \beta = -\frac{1}{2}\theta^2;$

where θ is an imaginary third root of unity, while γ is not determined at all by this condition.

We have, therefore, the following results due to DARBOUX. *The most general quadric Q which has contact of the second order with S at P contains three arbitrary constants. It is not possible to dispose of these in such a way as to have the three tangents of the triple point of the curve of intersection of S and Q coincide with an arbitrary line, tangent to S at P. If the three tangents of the triple point coincide at all, they will coincide with one of the three tangents of S given by the equation,*

$$x^3 + y^3 = 0. \tag{4}$$

Moreover, to each of these three tangents there corresponds a single infinity of quadrics of this kind.

These three tangents are called tangents of quadric osculation. They have recently been obtained by SEGRE* from a different point of view.

Obviously we may obtain in this way a family of curves on the surface S, three of which pass through every regular point. These curves DARBOUX denotes as the *lines of quadric osculation.*

§ 2. *The osculating Steiner surface.*

Let

$$x = \frac{\phi_1(\lambda, \mu)}{\phi_4(\lambda, \mu)}, \qquad y = \frac{\phi_2(\lambda, \mu)}{\phi_4(\lambda, \mu)}, \qquad z = \frac{\phi_3(\lambda, \mu)}{\phi_4(\lambda, \mu)}. \tag{5}$$

where

$$\phi_k = a_k\lambda^2 + 2h_k\lambda\mu + b_k\mu^2 + 2g_k\lambda + 2f_k\mu + c_k, \quad (k = 1, 2, 3, 4), \tag{6}$$

be the equations of a Steiner surface, λ and μ being the parameters which determine one of its points. It is well known † that any Steiner surface may be so represented. Since, moreover, the most general Steiner surface depends upon fifteen arbitrary constants and since the requirement that one surface shall have contact of the fourth order with a given one is equivalent to fifteen conditions, we may expect to be able to determine one, or at most a finite number of Steiner surfaces which have contact of the fourth order with S at P. We shall find that a unique surface of this kind exists.

The equations (5) and (6) may be simplified somewhat without loss of generality. Let us assume first that the parameters of the point $x = y = z = 0$ are $\lambda = \mu = 0$. Then we may put $c_1 = c_2 = c_3 = 0$ and $c_4 = 1$. Moreover, let

$$g_1 f_2 - g_2 f_1 \neq 0. \tag{7}$$

* C. SEGRE. *Complementi alla teoria delle tangenti coniugate di una superficie.* Rendiconti della R. Accademia dei Lincei, vol. 17 (1908), series 5ª, 2° sem. fasc. 9°. Seduta dell' 8 novembre 1908.

† According to WEIERSTRASS; cf. CLEBSCH in Crelle's Journal, vol. 67 (1867).

Then the quantities

$$\alpha = 2g_1\lambda + 2f_1\mu, \qquad \beta = 2g_2\lambda + 2f_2\mu,$$

may be introduced as new parameters in place of λ and μ and it will still be true that for $\alpha = \beta = 0$, x, y and z will all vanish. By these assumptions equations (6) reduce to

$$\begin{aligned}
\phi_1 &= \alpha + A_1\alpha^2 + 2H_1\alpha\beta + B_1\beta^2,\\
\phi_2 &= \beta + A_2\alpha^2 + 2H_2\alpha\beta + B_2\beta^2,\\
\phi_3 &= 2G_3\alpha + 2F_3\beta + A_3\alpha^2 + 2H_3\alpha\beta + B_3\beta^2,\\
\phi_4 &= 1 + 2G_4\alpha + 2F_4\beta + A_4\alpha^2 + 2H_4\alpha\beta + B_4\beta^2,
\end{aligned} \tag{8}$$

where G_4 and F_4 may moreover be equated to zero. For if they are not zero the transformation of the parameters given by

$$\alpha = \frac{\alpha'}{1 - G_4\alpha' - F_4\beta'}, \qquad \beta = \frac{\beta'}{1 - G_4\alpha' - F_4\beta'},$$

will give rise to a system of the same form as (8) but with zeros in the place of G_4 and F_4.

If these simplified expressions be substituted for x, y and z in (1), and if the coefficients of all of the powers of α and β up to and including those of the fourth order be equated to zero, the resulting Steiner surface will have contact of the fourth order with S at P. The resulting homogeneous equations are

$$\begin{aligned}
\rho x_1 &= 576\alpha - 72I\alpha^2 + 72J\alpha\beta - 96\beta^2,\\
\rho x_2 &= 576\beta - 96\alpha^2 + 72I\alpha\beta - 72J\beta^2,\\
\rho x_3 &= 576\alpha\beta,\\
\rho x_4 &= 576 + (24J - 9I^2)\alpha^2 + (18IJ - 80)\alpha\beta + (24I - 9J^2)\beta^2.
\end{aligned} \tag{9}$$

In deducing these formulæ we assumed $g_1f_2 - g_2f_1 \neq 0$. Suppose

$$g_1f_2 - g_2f_1 = 0 \qquad \text{but} \qquad g_1f_3 - g_3f_1 \neq 0.$$

We may again assume $c_1 = c_2 = c_3 = 0$, $c_4 = 1$, $G_4 = F_4 = 0$. If we put

$$2g_1\lambda + 2f_1\mu = \alpha, \qquad 2g_3\lambda + 2f_3\mu = \beta,$$

we shall have, instead of (8), the system

$$\begin{aligned}
\phi_1 &= \alpha + A_1\alpha^2 + 2H_1\alpha\beta + B_1\beta^2,\\
\phi_2 &= k\alpha + A_2\alpha^2 + 2H_2\alpha\beta + B_2\beta^2,\\
\phi_3 &= \beta + A_3\alpha^2 + 2H_3\alpha\beta + B_3\beta^2,\\
\phi_4 &= 1 + A_4\alpha^2 + 2H_4\alpha\beta + B_4\beta^2.
\end{aligned} \tag{8a}$$

But it is easy to see that a surface whose equations are of this form cannot have contact even of the first order with S at P. In fact, its tangent plane at the point $P(\alpha = \beta = 0)$ would be

$$kx - y = 0$$

instead of $z = 0$. Finally, both of the determinants $g_1 f_2 - g_2 f_1$ and $g_1 f_3 - g_3 f_1$ cannot vanish simultaneously since the point P is supposed to be a regular point for the surface S and therefore also for the osculating Steiner surface. We may therefore state the following theorem: *A unique Roman surface of Steiner can be determined which has contact of the fourth order with a given non-ruled analytic surface at one of its regular points. Referred to the canonical tetrahedron the parametric equations of this Steiner surface are given by equations* (9).

We shall refrain from developing in detail the geometric relations between this Steiner surface and the elements of the canonical tetrahedron, all of which may be expressed in terms of the invariants I and J.

§ 3. *Osculating conics of plane sections.*

Let the behavior of the surface S in the vicinity of the regular point P be expressed by its canonical development

$$\begin{aligned} z = xy + \tfrac{1}{6}(x^3 + y^3) + \tfrac{1}{24}(Ix^4 + Jy^4) + \frac{1}{5!}(K_0 x^5 + 5K_1 x^4 y + 10K_2 x^3 y^2 \\ + 10K_3 x^2 y^3 + 5K_4 xy^4 + K_5 y^5) + \frac{1}{6!}(L_0 x^6 + 6L_1 x^5 y + 15L_2 x^4 y^2 \\ + 20L_3 x^3 y^3 + 15L_4 x^2 y^4 + 6L_5 xy^5 + L_6 y^6) + \cdots, \end{aligned} \tag{10}$$

the terms up to those of the sixth order being explicitly written down. Consider the intersection of this surface with the plane

$$z = \lambda x + \mu y, \qquad \lambda = \frac{\lambda_1}{\lambda_3}, \qquad \mu = \frac{\lambda_2}{\lambda_3}. \tag{11}$$

The projection of this plane section upon the xy plane will be given by the development

$$\begin{aligned} \lambda x + \mu y - xy - \frac{1}{3!}(x^3 + y^3) - \frac{1}{4!}(Ix^4 + Jy^4) - \frac{1}{5!}(K_0 x^5 + 5K_1 x^4 y \\ + 10K_2 x^3 y^2 + 10K_3 x^2 y^3 + 5K_4 xy^4 + K_5 y^5) - \frac{1}{6!}(L_0 x^6 + 6L_1 x^5 y \\ + 15L_2 x^4 y^2 + 20L_3 x^3 y^3 + 15L_4 x^2 y^4 + 6L_5 xy^5 + L_6 y^6) - \cdots = 0. \end{aligned} \tag{12}$$

If the secant plane does not contain the z axis, the projection of the osculating

conic of the plane section will be the osculating conic of the curve (12) in the xy plane: and conversely, the quadric cone whose vertex is the center of projection (the point at infinity of the z axis) and whose directrix is the conic osculating the curve (12) at the origin, will determine as its intersection with the plane (11) the osculating conic of the plane section considered. To be sure, the osculating conics of a plane section whose plane contains the z axis escapes this method of treatment. It may be obtained by choosing a different center of projection, or more simply by considering it as the limit of a sequence of conics determined by a sequence of planes which approach coincidence with the plane considered.

Let

$$ax^2 + 2hxy + by^2 + 2gx + 2fy = 0 \tag{13}$$

be the osculating conic of (12). Then this equation (13) must be satisfied to terms of the fourth order inclusive if the value of y as function of x be substituted into it from (12). If $\mu \neq 0$ we may always assume that (12) is capable of a solution of the form

$$y = kx + lx^2 + mx^3 + nx^4 + px^5 + qx^6 + \cdots. \tag{14}$$

If μ vanishes, a similar expression may be found for x in terms of y, unless λ also is equal to zero. But there is no need of considering these cases separately as it will become evident how the formulae will have to be modified for $\mu = 0$. Finally if $\lambda = \mu = 0$, i. e., if the secant plane is the tangent plane, the osculating conic obviously degenerates and becomes indeterminate. Let us assume $\mu \neq 0$. The substitution of (14) in (12) will give

$$k = -\frac{\lambda}{\mu}, \quad l = -\frac{\lambda}{\mu^2}, \quad m = \frac{1}{6\mu^4}(-\lambda^3 - 6\lambda\mu + \mu^3),$$

$$n = \frac{1}{24\mu^5}[J\lambda^4 - 16\lambda^3 - 24\lambda\mu + 4\mu^3 + I\mu^4], \tag{15}$$

$$\mu p = n + \tfrac{1}{2}k^2m + \tfrac{1}{2}kl^2 + \tfrac{1}{6}Jk^3l + \tfrac{1}{1\,2\,0}(K, k),$$

$$\mu q = p + klm + \tfrac{1}{2}k^2n + \tfrac{1}{6}l^3 + \tfrac{1}{2\,4}J(4k^3m + 6k^2l^2) + \tfrac{1}{1\,2\,0}(K, k)'l + \tfrac{1}{7\,2\,0}(L, k),$$

where

$$(K, k) = K_0 + 5K_1k + 10K_2k^2 + 10K_3k^3 + 5K_4k^4 + K_5k^5,$$

$$(K, k)' = 5K_1 + 20K_2k + 30K_3k^2 + 20K_4k^3 + 5K_5k^4,$$

$$(L, k) = L_0 + 6L_1k + 15L_2k^2 + 20L_3k^3 + 15L_4k^4 + 6L_5k^5 + L_6k^6.$$

If we substitute (14) into equation (13) and equate to zero the coefficients of x, x^2, x^3, x^4 we obtain the conditions necessary for the determination of the osculating conic. Upon solution these give

$$
\begin{aligned}
&a = 2(-l^4 + kml^2 - k^2ln + k^2m^2) = \lambda^2(\alpha - 12\beta - 12\lambda\mu^4),\\
&b = 2(m^2 - ln) = \mu^2(\alpha - 12\beta - 12\lambda^4\mu),\\
&h = -ml^2 + 2kln - 2km^2 = \lambda\mu(\alpha - 18\beta + 36\lambda^2\mu^2),\\
&f = l^3 = -36\lambda^3\mu^4, \qquad g = -kl^3 = -36\lambda^4\mu^3,
\end{aligned} \tag{17}
$$

where

$$
\begin{aligned}
&\alpha = 2(\lambda^3 - \mu^3)^2 + 3\lambda\mu(J\lambda^4 + I\mu^4),\\
&\beta = \lambda\mu(\lambda^3 + \mu^3).
\end{aligned} \tag{18}
$$

If $\mu = 0$, these formulæ give as the osculating conic of a plane section through one of the asymptotic tangents that tangent counted twice. This same result is obtained if the development in powers of y be employed which exists in place of (14) when $\mu = 0$, $\lambda \neq 0$. If both λ and μ vanish, the secant plane is tangent to the surface and the osculating conic becomes indeterminate. It consists of one of the asymptotic tangents and any other tangent of the surface at the given point. The pair of asymptotic tangents in that case constitutes a degenerate hyperosculating conic.

Let us examine the locus of the osculating conics corresponding to a fixed tangent, i. e., corresponding to those secant planes for which

$$\frac{\lambda}{\mu} = c \tag{19}$$

is a constant. Put

$$A = 2(c^3 - 1)^2 + 3c(Jc^4 + I), \qquad B = c(c^3 + 1). \tag{20}$$

If we divide by the common factor μ^6, the coefficients of the equation of the projection of the osculating conic become

$$
\begin{aligned}
&a = c^2\mu[A\mu - 12(B + c)], \qquad b = \mu[A\mu - 12(B + c^4)],\\
&h = c(A\mu^2 - 18B\mu + 36c^2), \qquad f = -36c^3\mu, \qquad g = -36c^4\mu.
\end{aligned} \tag{21}
$$

The locus of these conics is found by eliminating μ between the equations

$$ax^2 + 2hxy + by^2 + 2gx + 2fy = 0, \qquad z = \mu(cx + y), \tag{22}$$

where, a, h, b, g, f are given by (21). The result is

$$Az^2 - 72c^3z - 12(c^2 + Bc)xz - 12(c^4 + B)yz + 72c^3xy = 0, \tag{23}$$

so that the required locus is a quadric surface, which, as may be easily verified, is never a cone unless $c = 0$ or ∞, in which cases it degenerates into the tangent plane counted twice. We have thus proved the theorem of MOUTARD for the case of a non-ruled surface. Since (23) is of the sixth degree in c, we see that six quadrics of this kind pass through an arbitrary point of space which

is not in the tangent plane. All of these quadrics have contact of the second-order with S at the origin, as they are included in the form (2). We may express these results as follows. *Consider the section of a non-ruled surface S made by a plane passing through one of its regular points P. The conic which has fourth-order contact with this plane section at P generates a quadric surface if the plane turns about a fixed tangent of the surface at P as axis. Thus there corresponds a unique quadric surface to every tangent of the surface at P. Each of these quadrics has second order contact with S, and six of them pass through an arbitrary point of space which is not in the tangent plane. Leaving aside the case when one of the quadrics degenerates into the tangent plane counted twice, there exists no cone among them.*

We proceed to complete these considerations by studying the plane section of the surface S made by a pencil of planes whose axis passes through P, but is not a tangent of the surface.

Let

$$z = \lambda x + \mu y, \qquad \text{or} \qquad \lambda_3 z = \lambda_1 x + \lambda_2 y \tag{24}$$

be the equation of a plane which contains the point whose coördinates in the non-homogeneous canonical system are α, β, γ. Then we shall have the relation

$$\gamma = \lambda\alpha + \mu\beta, \qquad \gamma \neq 0, \tag{24a}$$

between λ and μ, where of course only the ratios of $\alpha : \beta : \gamma$ are of importance. This plane intersects the tangent plane in the straight line

$$z = 0, \qquad \lambda x + \mu y = 0, \qquad \text{where} \qquad \lambda\alpha + \mu\beta = \gamma, \tag{25}$$

and the osculating conic of its intersection with S will be the intersection of the plane (24) with the MOUTARD quadric which belongs to the tangent (25). We may write the equation of the plane (24) as follows

$$z = \lambda x + \frac{\gamma - \lambda\alpha}{\beta} y. \tag{26}$$

The MOUTARD quadric of the tangent of S in which (26) intersects the tangent plane is given by (23) if the value of c which occurs in that equation be equated to

$$c = \frac{\lambda}{\mu} = \frac{\beta\lambda}{\gamma - \lambda\alpha}. \tag{27}$$

The intersection of (26) and this quadric is the osculating conic of the plane section considered. The elimination of λ between (23), (26), and (27) will give the equation of the locus of the osculating conics of the plane sections of S determined by the pencil of planes considered.

We find

$$c = \frac{\gamma y - \beta z}{-\gamma x + \alpha z} \tag{28}$$

which may be substituted in (23) to obtain the required locus, which is of the eighth order. It will be convenient however to introduce a new system of non-homogeneous coördinates by putting

$$\xi = -\gamma x + \alpha z, \qquad \eta = \gamma y - \beta z, \qquad \zeta = z, \tag{29a}$$

or

$$x = \frac{1}{\gamma}(-\xi + \alpha\zeta), \qquad y = \frac{1}{\gamma}(\eta + \beta\zeta), \qquad z = \zeta, \tag{29b}$$

a transformation which is equivalent to choosing as ζ axis the axis of the given pencil of planes without changing the x or y axes.

If this be done, the equation of the required locus becomes

$$\begin{aligned}\zeta^2[&2\gamma^2\xi^6 + 3\gamma(\gamma J - 4\alpha)\xi^5\eta - 24\beta\gamma\xi^4\eta^2 + 4(18\alpha\beta - \gamma^2)\xi^3\eta^3 \\ &- 24\alpha\gamma\xi^2\eta^4 + 3\gamma(\gamma I - 4\beta)\xi\eta^5 + 2\gamma^2\eta^6] + \zeta[-72\gamma^2\xi^3\eta^3 \\ &+ 12\gamma\xi^6\eta - 24(\gamma + 3\beta)\xi^4\eta^3 + 24(\gamma + 3\alpha)\xi^3\eta^4 - 12\gamma\xi\eta^6] \\ &- 27\xi^4\eta^4 = 0.\end{aligned} \tag{30}$$

It may be verified that this surface of the eighth order, S_8, has a septuple point at the origin and is therefore rational. The ζ axis is a sextuple line, while the ξ and η axes are both double lines of the surface. The two planes that are tangent to the two sheets of S_8 which intersect along the ξ axis are fixed, i. e., they do not vary with the point of contact. The same is true of the two tangent planes which belong to any point of the η axis. One of these tangent planes moreover is the plane $\zeta = 0$, tangent to S at P. The other two planes containing the ξ and η axis respectively are

$$\gamma\zeta + 6\eta = 0 \qquad \text{and} \qquad \gamma\zeta - 6\xi = 0, \tag{31a}$$

or returning to the canonical system of coördinates

$$6\gamma y + (\gamma - 6\beta)z = 0, \qquad 6\gamma x + (\gamma - 6\alpha)z = 0. \tag{31b}$$

Since, moreover, the intersection of the $\xi\eta$ plane with S_8 consists of the ξ and η axis each counted four times, the $\xi\eta$ plane is a stationary tangent plane of S_8 along the entire ξ and η axes. The surface S_8 has no other nodal points than those indicated.

Referred to the canonical tetrahedron, (30) is homogeneous with respect to α, β, γ, as it should be.

We are in a position to express more clearly than formerly the significance of the canonical system of coördinates. The tetrahedron of reference requires no further discussion, but the unit point of the system was defined in a somewhat indirect way.* Let us now consider the surface S_8 whose axis is the z axis. Its equation is obtained from (30) if we put the ratios of $\alpha:\gamma$ and $\beta:\gamma$ equal to zero. The two planes (31b) then reduce to

$$6y + z = 0, \qquad 6x + z = 0. \tag{32}$$

Their line of intersection intersects the canonical quadric of the point P of the surface S,

$$z = xy$$

in the origin and the further point

$$x = -6, \qquad y = -6, \qquad z = 36.$$

Let us speak of the surface S_8 which is determined by any line l through P as a surface of osculating conics; let l be called its axis and let the two planes (31b) be called its singular tangent planes. We find the following result.

The unit point of the canonical system of coördinates is chosen in such a way that the second point which the singular tangent planes of that surface of osculating conics, whose axis is the directrix of the second kind, have in common with the canonical quadric shall have as its coördinates the values

$$x = -6, \qquad y = -6, \qquad z = 36.$$

If (13) is the equation of the projection of the osculating conic of one of the plane sections of the surface S, its coefficients are given by (17) or (21). If the development (14) for y be substituted in the left member of (13), the resulting power-series will have zero coefficients for its first five terms. Let C_5 and C_6 denote the coefficients of x^5 and x^6 in this power series. Then

$$\begin{aligned} C_5 &= 2fp + 2hn + 2(kn + lm)b, \\ C_6 &= 2fq + 2hp + (2kp + 2ln + m^2)b. \end{aligned} \tag{33}$$

We find from (15)

$$p = \frac{1}{120\mu^7}[(K, k)\mu^6 - 10\lambda^5 + 25J\lambda^4\mu + 10\lambda^2\mu^3 + 5I\mu^5 - 200\lambda^3\mu + 20\mu^4 - 120\lambda\mu^2],$$

$$\begin{aligned} q = \frac{1}{720\mu^8}[&(L, k)\mu^7 + 35J\lambda^6 - 20J\lambda^3\mu^3 + 15I\lambda^2\mu^4 - 6(K, k)'\lambda\mu^5 + 6(K, k)\mu^6 \\ &- 420\lambda^5 + 450J\lambda^4\mu + 240\lambda^2\mu^3 + 30I\mu^5 - 2400\lambda^3\mu + 120\mu^4 - 720\lambda\mu^2]. \end{aligned}$$

Consequently we shall have

*Second memoir. These Transactions, vol. 9 (1908), p. 110.

$$k=-c, \qquad l=-\frac{c}{\mu}, \qquad m=\frac{1}{6\mu^2}[(1-c^3)\mu-6c],$$

$$n=\frac{1}{24\mu^3}[(Jc^4+I)\mu^2+4(1-4c^3)\mu-24c],$$

$$(35)\quad p=\frac{1}{120\mu^4}[(K,k)\mu^3+(-10c^5+25Jc^4+10c^2+5I)\mu^2+5(-40c^3+4)\mu-120c],$$

$$q=\frac{1}{720\mu^5}[(L,k)\mu^4+\{35Jc^6-20Jc^3+15Ic^2-6(K,k)'c+6(K,k)\}\mu^3$$
$$+(-420c^5+450Jc^4+240c^2+30I)\mu^2+(-2400c^3+120)\mu-720c].$$

It will be convenient to make use of the relation

$$(36)\qquad h+kb=6(c^5-c^2)\mu+36c^3$$

in computing C_6.

We find

$$\Lambda=\frac{30\mu^2}{c}C_5=[20(c^3-1)^3-18c^2(K,-c)+45c(c^3-1)(I+Jc^4)]\mu^3$$
$$-180c[c^6-1+c(Jc^4-I)]\mu+360c^2(c^3-1),$$

$$(37)\quad \mathrm{M}=720\mu^4C_6=[-72(L,-c)c^3+72(K,-c)(c^3-1)c^2-60c(Jc^4+I)A$$
$$+20(c^3-1)^2A]\mu^3+[432(K,-c)'c^4+4320c^5(Jc^4+I)-1080c^2(Jc^4+I)$$
$$+1200c(c^9-1)-3600c^4(c^3-1)]\mu^2+[-15120Jc^7+6480Ic^3-12960c^8$$
$$-8640c^2(c^3-1)]\mu+25920c^6-17280c^3.$$

For a given value of c, there are two values of μ for which Λ becomes equal to zero. In other words, *through every tangent of the surface there can be passed two planes in general different from the tangent plane, such that their intersections with the surface shall be hyperosculated by their osculating conics.* It is easy to verify that through every line which contains the given point P of the surface S, and which is not tangent to S at P, there pass nine such planes. Consequently, all of the planes whose intersections with the surface S are hyperosculated by their osculating conics at the point P, generate a cone of the ninth class, which has the tangent plane as a septuple plane. The elements, along which this cone touches the tangent plane, are the asymptotic tangents, each of which counts as two, and the three Darboux tangents of quadric osculation.

Among the plane sections just considered, for which $C_5=0$, there will be some for which C_6 also vanishes. Their osculating conics will have seven consecutive points in common with them. They are determined by the simultaneous equations $\Lambda=\mathrm{M}=0$. It seems desirable, however, to replace the

equation $\mathrm{M} = 0$ which is not symmetrical with respect to I and J, by the equation

$$\mathrm{N} = \mathrm{M} - 60c\Lambda = 0$$

which is symmetrical. Put

$$\begin{aligned}
D &= 20(c^3-1)^3 - 18c^2(K,-c) + 45c(c^3-1)(I+Jc^4),\\
E &= -180\,[c^6 - 1 + c(Jc^4 - I)], \qquad F = 360(c^3-1),\\
D' &= -72(L,-c)c^3 + 72(K,-c)c^2(c^3-1) - 60c(Jc^4+I)A + 20(c^3-1)^2A,\\
E' &= 432(K,-c)'c^3 + 1080(K,-c)c^2 + 1620c(c^3+1)(Jc^4+I),\\
F' &= -4320c(Jc^4+I) - 2160c^6 - 8640c^3 - 2160,\\
G' &= 4320(c^3+1).
\end{aligned} \tag{38}$$

Then the two conditions for seven-pointic conics become

$$\begin{aligned}
D\mu^2 + Ec\mu + Fc^2 &= 0,\\
D'\mu^3 + E'c\mu^2 + F'c^2\mu + G'c^3 &= 0.
\end{aligned} \tag{39}$$

Let us put

$$\mu = \frac{1}{m} \tag{40}$$

so that (39) become

$$\begin{aligned}
D + Ecm + Fc^2m^2 &= 0,\\
D' + E'cm + F'c^2m^2 + G'c^3m^3 &= 0.
\end{aligned} \tag{41}$$

We may eliminate cm between these two equations by SYLVESTER's dialytic method. The resultant is

$$\Delta = \begin{vmatrix} D & E & F & 0 & 0\\ 0 & D & E & F & 0\\ 0 & 0 & D & E & F\\ D' & E' & F' & G' & 0\\ 0 & D' & E' & F' & G' \end{vmatrix} = 0, \tag{42}$$

which is certainly not of degree higher than 33 in c, as may be seen at once by noting that D, E, F are of degrees 9, 6, 3 and D', E', F', G' of degrees 12, 9, 6, 3 respectively. We shall see, however, that Δ is really only of the thirtieth degree in c, and that it has c^3 as a factor. So that after division by this factor, there will remain an irreducible equation of the twenty-seventh degree.

We shall show first that Δ contains c^3 as a factor. Let us put $cm = u$ in (41) and interpret u and c as cartesian coördinates. The two curves (41) will have the point $c = 0$, $u = \frac{1}{6}$ in common, as is easily verified. In fact for $c = 0$, both equations are satisfied by $u = \frac{1}{6}$ and by no other common value of u. Let us put

$$u - \tfrac{1}{6} = v.$$

The two equations become

$$d + ev + fv^2 = 0, \qquad d' + e'v + f'v^2 + g'v^3 = 0, \tag{43}$$

where

$$d = D + \tfrac{1}{6}E + \tfrac{1}{36}F, \qquad e = E + \tfrac{1}{3}F, \qquad f = F,$$

$$d' = D' + \tfrac{1}{6}E' + \tfrac{1}{36}F' + \tfrac{1}{216}G', \qquad e' = E' + \tfrac{1}{3}F' + \tfrac{1}{12}G', \tag{44}$$

$$f' = F' + \tfrac{1}{2}G', \qquad g' = G'.$$

If these coefficients be developed with respect to powers of c, we shall find

$$d = -15Ic - 18K_0c^2 + (70 + 90K_1)c^3 + \cdots,$$

$$e = 60 + 180Ic + 120c^3 + \cdots, \qquad f = -360 + 360c^3,$$

$$d' = 90Ic + (108K_0 - 180I^2)c^2 - (72L_0 + 180K_1 + 300)c^3 + \cdots, \tag{45}$$

$$e' = 360 + 180Ic + 1080K_0c^2 - (3240K_1 - 2520)c^3 + \cdots,$$

$$f' = -4320Ic - 6480c^3, \qquad g' = 4320 + 4320c^3.$$

We may now obtain from either equation of (43) a development for v in powers of c, and we shall find that these two developments agree up to the terms of the second order inclusive. These common terms of the developments are

$$v = \tfrac{1}{4}Ic + \tfrac{1}{120}(36K_0 - 45I^2)c^2 \cdots;$$

the terms of the third order in the two developments do not agree in general. We see, therefore, that the two curves (41) have contact of second order at the point $c = 0$, $u = \frac{1}{6}$, consequently this point counts as three among the intersections of the two curves, and Δ must contain c^3 as a factor. In the same way, it may be shown, and moreover it can be seen without further computation, that $c = \infty$ must be a triple root of the equation $\Delta = 0$, i. e., the coefficients of the three highest powers of c in Δ must vanish. The eliminant of the equations (41), therefore, reduces to the twenty-seventh degree. The three roots $c = 0$ and $c = \infty$, which have just been removed from (42), obviously do not give proper solutions of the problem under consideration. The other twenty-seven do. In fact, if the surface considered were a cubic, a conic having seven-pointic contact with it would have to be entirely upon the surface. But owing to the existence of twenty-seven straight lines upon a cubic surface, there exists upon it a family of conics, of which twenty-seven pass through every point. Since in this special case twenty-seven seven-pointic conics actually exist, we see that our eliminant of the twenty-seventh order contains no factors strange to the problem. We conclude therefore that the following theorem is true: *At a regular point of a non-ruled surface twenty-seven conics may be constructed having contact of the sixth order with it.*

The discriminant of the equation $\Lambda = 0$, after removing the extraneous factor c^2, is of the twelfth degree in c. *Consequently there are twelve tangents through a regular point of a non-ruled surface for which the two hyperosculating conics coincide.*

§ 4. *Osculating and hyperosculating conics at a regular point of a ruled surface.*

If the surface S under consideration is a ruled surface the development (10) breaks down. We may write in that case

$$(46) \qquad z = xy + x^3 + K_0x^5 + K_1x^4y + K_2x^3y^2 + \frac{1}{6!}L(x, y) + \cdots,^*$$

where

$$(47) \qquad L(x, y) = L_0x^6 + 6L_1x^5y + 15L_2x^4y^2 + 20L_3x^3y^3 + 15L_4x^2y^4,$$

the terms involving y^5 and y^6 being necessarily absent. More generally, if we denote by u_n the terms of the nth degree in the development (46), the coefficients of y^n and y^{n-1} in u_n must be equal to zero. In fact, if the coefficients of y^n were not zero, the line $x = z = 0$ which is a generator of the ruled surface could not be upon it. If the coefficient of y^{n-1} were not equal to zero, the planes which contain this generator g, each of which is tangent to the surface at one of the points of g, could not be projectively related to their points of contact. (CHASLES's correlation.)

As before, let us cut the surface by a plane

$$z = \lambda x + \mu y,$$

and consider the projection of the curve of intersection

$$\lambda x + \mu y - xy - x^3 - K_0x^5 - K_1x^4y - K_2x^3y^2 - \frac{1}{6!}L(x, y) - \cdots = 0.$$

We shall find

$$y = kx + lx^2 + mx^3 + nx^4 + px^5 + qx^6 + \cdots.$$

where

$$k = -\frac{\lambda}{\mu}, \qquad l = \frac{\lambda}{\mu^2}, \qquad m = \frac{\mu^2 - \lambda}{\mu^3}, \qquad n = \frac{\mu^2 - \lambda}{\mu^4},$$

$$(48) \qquad p = \frac{1}{\mu^5}(K_0\mu^4 + K_2\lambda^2\mu^2 - K_1\lambda\mu^3 + \mu^2 - \lambda),$$

$$q = \frac{1}{\mu^6}\left[\frac{1}{720}\mu^5\left(L, -\frac{\lambda}{\mu}\right) + K_0\mu^4 - 2K_1\lambda\mu^3 + 3K_2\lambda^2\mu^2 + \mu^2 - \lambda\right],$$

* Third Memoir. These Transactions, vol. 9 (1908), p. 299.

and where $(L, -\lambda/\mu)$ denotes $L(1, -\lambda/\mu)$. The projection of the osculating conic becomes

$$ax^2 + 2hxy + by^2 + 2gx + 2fy = 0 \tag{49}$$

where

$$a = 2c^2\mu(\mu - 2c), \quad b = 2\mu(\mu - c), \quad h = c(\mu - c)(2\mu - c), \tag{50}$$
$$f = -c^3\mu, \qquad g = -c^4\mu, \qquad c = \lambda/\mu.$$

The locus of the osculating conics of all the sections whose planes pass through a fixed tangent, $(\lambda/\mu = c)$, *is again a quadric surface, viz.;*

$$z^2 - (2c^2x + cy + c^3)z + c^3xy = 0. \tag{51}$$

We see that *three of these quadrics pass through any point of space which is not in the tangent plane*, while in the case of a non-ruled surface there were six. As in the general case, none of these quadrics is a cone except in the case when it degenerates into the tangent plane counted twice.

Consider the plane sections of S whose planes have in common a fixed line l through the given point P when l is *not* a tangent of the surface. As in the general case, we denote by α, β, γ the coördinates of any point of this line l and put

$$\xi = -\gamma x + \alpha z, \qquad \eta = \gamma y - \beta z, \qquad \zeta = z, \qquad \gamma \neq 0.$$

The locus of the osculating conics of these plane sections is the quintic surface

$$\zeta^2[\gamma^2\xi^3 - 2\alpha\gamma\xi\eta^2 - \beta\gamma\xi^2\eta + \alpha\beta\eta^3] + \zeta[\gamma\xi^2\eta^2 - \gamma^2\eta^3 - \beta\xi\eta^3 + \alpha\eta^4] - \xi\eta^4 = 0, \tag{52}$$

which has the origin as a quadruple point and is therefore rational, which has the given line l as a triple line, and which has the asymptotic tangent t of S at P as a double line. Moreover, the two sheets of the quintic (52) *which have the points of t in common are tangent to each other and to the tangent plane of S throughout the entire extent of t.*

We find in this case

$$C_5 = \frac{2c}{\mu^2}[-(2 + K_0c^2 - K_1c^3 + K_2c^4)\mu^2 + 3c\mu - c^2],$$

$$C_6 = \frac{1}{\mu^3}[\{2 - 2K_0c^2 + 2K_1c^3 - 2K_2c^4 - \tfrac{1}{3}\tfrac{1}{6}\tfrac{0}{0}c^3(L, -c)\}\mu^3 \tag{53}$$
$$+ (-10c + 2K_1c^4 - 4K_2c^5)\mu^2 + 12c^2\mu - 4c^3].$$

Consequently *there exist again two sections through every tangent, whose osculating conics hyperosculate them. There are four tangents through P such that these hyperosculating conics coincide.* The conics which have seven-point contact with the ruled surface at P are obtained by equating C_5 and C_6 simultaneously to zero. Put again

$$\mu = \frac{c}{u},$$

so that the conditions $C_5 = C_6 = 0$ become

$$\begin{aligned} D - 3u + u^2 &= 0, \\ D' + E'u + 12u^2 - 4u^3 &= 0, \end{aligned} \tag{54}$$

where

$$\begin{aligned} D &= 2 + K_0 c^2 - K_1 c^3 + K_2 c^4, \\ D' &= 2 - 2K_0 c^2 + 2K_1 c^3 - 2K_2 c^4 - \frac{c^3}{360}(L, -c), \\ E' &= -10 + 2K_1 c^3 - 4K_2 c^4. \end{aligned} \tag{55}$$

The resultant of (54) is

$$D(E' + 4D)^2 + 3D'(E' + 4D) + D'^2 = 0, \tag{56}$$

which is of the fourteenth degree, but may be easily shown to contain c^2 and no higher power of c as a factor if

$$720K_1 - L_0 \neq 0. \tag{57}$$

We may therefore state our result as follows: *At a regular point of a ruled surface there can be constructed, in general, twelve conics having contact of the sixth order with the surface.*

If the surface is a developable, the expansion of its equation in the vicinity of an ordinary point may be written in the form *

$$z = x^2 + Kx^3y + \frac{1}{6!}(L_0x^6 + 6L_1x^5y + 15L_2x^4y^2 + 20L_3x^3y^3 + 15L_4x^2y^4) + \cdots, \tag{58}$$

where K may be supposed to be equal to unity or zero, according as a certain differential invariant of the fourth order is or is not different from zero. In the latter case however there will, in general, be present in the development also one of the terms of the fifth order. We shall find incidentally the significance of the vanishing of this differential invariant of the fourth order as a consequence of the considerations which follow.

If we maintain the same notations as in the other cases, we shall find, assuming that the secant plane does not pass through the generator, $(\mu \neq 0)$,

$$\begin{aligned} k &= -\frac{\lambda}{\mu}, \quad l = \frac{1}{\mu}, \quad m = 0, \quad n = -\frac{\lambda K}{\mu^2}, \\ p &= \frac{K}{\mu^2}, \quad q = \frac{1}{720\mu}(L, k), \end{aligned} \tag{59}$$

whence we obtain

$$a = 2(Kc^3\mu^2 - 1), \quad b = 2Kc\mu^2, \quad h = 2Kc^2\mu^2, \quad f = \mu, \quad g = c\mu \tag{60}$$

* This is essentially the form of development given by DARBOUX. But this may be further simplified in the terms of the sixth order and the geometry of the transformation suggests a number of questions which will be left for a future occasion.

as the coefficients of the equation of the projection of the osculating conic of the section of the developable made by the plane

$$z = \lambda x + \mu y,$$

if λ/μ be denoted by c. *The locus of the osculating conics, as the secant plane turns around a fixed tangent t determined by the ratio $\lambda/\mu = c$, is again a quadric surface*

$$cKz^2 - x^2 + z = 0, \tag{61}$$

which, however, in this case, is a cone, and its vertex is a point on the y axis, which is independent of the value of c. The geometrical significance of the y axis in the development (58) is therefore apparent. It is of interest to notice that we obtain a single infinity of curves on the developable defined projectively, if we associate with each of the points of the surface as direction of progress the line which joins it to the vertex of the associated cones.

If we consider the osculating conics of the sections of the developable S that are made by a plane turning around a fixed line l which passes through the point P of the surface, but which is not tangent to S at P, we shall find as its locus a ruled surface of the third order (a Cayley cubic scroll) if K is different from zero.

If K vanishes, this surface degenerates into a plane and a quadric cone. This, then, is the significance of the vanishing of the differential invariant of the fourth order which was mentioned above.

We find

$$\tfrac{1}{2} C_5 = \frac{K}{\mu}(1 - 4kc^3\mu^2), \qquad C_6 = \tfrac{1}{3}\tfrac{1}{6}\overline{0}(L, -c) - 4Kc^2. \tag{62}$$

Consequently *there will be*, as in the general case, *two plane sections through a fixed tangent whose osculating conics hyperosculate it, if $K \neq 0$.* Since the condition $C_6 = 0$ is of the fourth degree in c, and to each of these values the condition $C_5 = 0$ makes correspond two values of μ, we see that *there will be eight conics which have contact of the sixth order with a developable at any one of its ordinary points.*

THE UNIVERSITY OF ILLINOIS,
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ON THE OSCULATING QUARTIC OF A PLANE CURVE*

BY

WILLIAM WELLS DENTON

The quartic curve which has contact of the thirteenth order with a given analytic plane curve at one of its points will be called the osculating quartic. In this paper, its equation will be found in an invariant form, referred to a triangle which has a simple projective relation to the given curve. The method which we shall employ is due to Professor WILCZYNSKI.†

Let there be given three linearly independent analytic functions of x,

$$y_\kappa = f_\kappa(x), \qquad (\kappa = 1, 2, 3). \tag{1}$$

They may be interpreted as the homogeneous coördinates of a point P_y in a plane. As x changes, P_y describes an analytic plane curve C_y. There exists a uniquely defined linear differential equation of the third order of which y_1, y_2, y_3 form a fundamental system. Let this be

$$y^{(3)} + p_1 y'' + p_2 y' + p_3 y = 0, \tag{2}$$

where p_1, p_2, p_3 are analytic functions of x. We may, therefore, speak of C_y as being an integral curve of (2). But this integral curve is not unique, for every projective transformation of C_y is likewise an integral curve of (2). Conversely, every integral curve of (2) is a projective transformation of C_y. Hence the properties of C_y determined by the coefficients (2) are common to all curves projectively equivalent to C_y, that is, they are projective properties.

The representation of a curve in the form (1), however, involves some arbitrary elements. In the first place, since the coördinates are homogeneous, a transformation of the form

$$\bar{y} = \lambda(x) y, \tag{3}$$

where $\lambda(x)$ is an arbitrary analytic function of x, does not affect the curve. Moreover, a transformation of the form

$$\bar{x} = \xi(x), \tag{4}$$

where $\xi(x)$ is an arbitrary analytic function of x, merely changes the para-

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† E. J. WILCZYNSKI, *Projective differential geometry of curves and ruled surfaces.* Teubner, Leipzig, 1906, Chapters II and III. We shall hereafter refer to this work as *Proj. Diff. Geom.*

metric representation, without changing the curve. Those combinations of the coefficients of (2) and of their derivatives which remain unchanged under all transformations of the form (3) and (4), the so-called invariants, therefore characterize the projective properties of the curve independently of its method of representation. If such an invariant function contains also y and its derivatives it is called a covariant.

It is convenient to consider the subgroup of the general group of transformations defined by equations (3) and (4), which is obtained by leaving the independent variable x fixed. The invariant functions for this subgroup are known as seminvariants and semicovariants. Of these we shall need *

$$z = y' + p_1 y, \qquad \rho = y'' + 2p_1 y' + p_2 y, \tag{5}$$

$$P_2 = p_2 - p_1^2 - p_1', \qquad P_3 = p_3 - 3p_1 p_2 + 2p_1^3 - p_1'', \tag{6}$$

as well as the following invariants: †

$$\theta_3 = P_3 - \tfrac{3}{2} P_2', \qquad \theta_8 = 6\theta_3\theta_3'' - 7(\theta_3')^2 - 27 P_2 \theta_3^2. \tag{7}$$

By a special transformation of the form (3) and (4), viz., ‡

$$\bar{y} = \frac{d\bar{x}}{dx} y, \qquad \bar{x} = \xi(x), \tag{8}$$

where $\xi(x)$ is determined by the equations:

$$\xi(x) = c_1 \int e^{\int \eta\, dx}\, dx + c_2, \qquad \eta' - \tfrac{1}{2}\eta^2 = \tfrac{3}{2} P_2, \tag{9}$$

in which c_1 and c_2 are arbitrary constants, equation (2) may be reduced to the LAGUERRE-FORSYTH canonical form:

$$\frac{d^3 \bar{y}}{d\bar{x}^3} + \bar{P}_3 \bar{y} = 0. \tag{10}$$

The invariant functions above mentioned have then the following exceptionally simple *canonical* forms:

$$\bar{z} = y', \qquad \bar{\rho} = y'', \qquad \bar{P}_2 = 0, \tag{11}$$

$$\bar{\theta}_3 = \bar{P}_3, \qquad \bar{\theta}_8 = 6\bar{P}_3\bar{P}_3'' - 7(\bar{P}_3')^2. \tag{12}$$

Hereafter, we shall assume the differential equation in this form, and, for convenience in writing, omit the dashes. We shall assume also that P_3, in this equation, is not identically equal to zero. This requires only that C_y be not a conic.

* *Proj. Diff. Geom.*, p. 58.

† Ibid., p. 59.

‡ Ibid., p. 25.

Let x_0 be the value of x which determines the point P_{y_0}, on the curve C_y. The point P_z, whose coördinates are

$$z_\kappa = \left[\frac{dy_\kappa}{dx}\right]_{x=x_0} \qquad (\kappa = 1, 2, 3),$$

is a point on the tangent to C_y at P_{y_0};* and, if P_{y_0} is not a point of inflection, the point P_ρ, whose coördinates are

$$\rho_\kappa = \left[\frac{d^2y_\kappa}{dx}\right]_{x=x_0} \qquad (\kappa = 1, 2, 3),$$

is not collinear with P_{y_0} and P_z; so that these points determine a non-degenerate triangle, semicovariantly related to the curve C_y. Let this be taken as a triangle of reference. We may choose the unit point of our system of homogeneous coördinates so that an expression of the form †

$$x_1 y(x_0) + x_2 z(x_0) + x_3 \rho(x_0) \tag{13}$$

will represent the point whose coördinates are precisely x_1, x_2, x_3.

Let $x = x_0$ be an ordinary point for the function P_3; and, for convenience in writing, let $x_0 = 0$, since this assumption involves no loss of generality. Then for values of $|x|$ sufficiently small, any solution of equation (10) may be expressed as a convergent power series,

$$G(x) = y(0) + y'(0)x + \frac{1}{2}y''(0)x^2 + \cdots + \frac{1}{14!}y^{(14)}(0)x^{14} + \cdots. \tag{14}$$

From equation (10), we find by direct differentiation:

$$\begin{aligned} y^{(3)} &= -ay, \qquad y^{(4)} = -a_1 y - az, \\ y^{(5)} &= -a_2 y - 2a_1 z - a\rho, \\ y^{(6)} &= -(a_3 - a^2)y - 3a_2 z - 3a_1\rho, \text{ etc.}, \end{aligned} \tag{15}$$

where we have put $y = y(0)$, $z = y'(0)$, $\rho = y''(0)$, $a = P_3$, and where a_m^n is an abbreviation for the nth power of the mth derivative of P_3 with respect to x. Putting $G(x)$ in the form (13), we find the following equations, which represent the curve C_y up to terms of the fourteenth order in the vicinity of the point P_{y_0}:

$$y_1 = 1 - \frac{a}{3!}x^3 - \frac{a_1}{4!}x^4 - \frac{a_2}{5!}x^5 - \frac{1}{6!}(a_3 - a^2)x^6 - \frac{1}{7!}(a_4 - 5aa_1)x^7$$

$$- \frac{1}{8!}(a_5 - 11aa_2 - 5a_1^2)x^8 - \frac{1}{9!}(a_6 - 21aa_3 - 21a_1a_2 + a^3)x^9$$

* *Proj. Diff. Geom.*, p. 54.

† *Proj. Diff. Geom.*, p. 61.

$$-\frac{1}{10!}(a_7-36aa_4-42a_1a_3-21a_2^2+12a^2a_1)x^{10}$$

$$-\frac{1}{11!}(a_8-57aa_5-78a_1a_4-84a_2a_3+39a_2a^2+45aa_1^2)x^{11}$$

$$-\frac{1}{12!}(a_9-85aa_6-135a_1a_5-162a_2a_4+105a^2a_3-84a_3^2 + 300aa_1a_2+45a_1^3-a^4)x^{12}$$

$$-\frac{1}{13!}(a_{10}-220a_1a_6-121aa_7-297a_2a_5-330a_3a_4+852aa_1a_3 + 246a^2a_4+435a_1^2a_2+516aa_2^2-22a^3a_1)x^{13}$$

$$-\frac{1}{14!}(a_{11}-166aa_8-341a_1a_7-517a_2a_6-627a_3a_5+519a^2a_5 - 330a_4^2+2124aa_1a_4+1287a_1^2a_3+3054aa_2a_3 + 1386a_1a_2^2-94a^3a_2-177a^2a_1^2)x^{14}$$

$$-\cdots,$$

$$y_2=x-\frac{a}{4!}x^4-\frac{2a_1}{5!}x^5-\frac{3a_2}{6!}x^6-\frac{1}{7!}(4a_3-a^2)x^7-\frac{1}{8!}(5a_4-7aa_1)x^8$$

$$-\frac{1}{9!}(6a_5-18aa_2-12a_1^2)x^9-\frac{1}{10!}(7a_6+a^3-39aa_3-63a_1a_2)x^{10}$$

$$-\frac{1}{11!}(8a_7-75aa_4-144a_1a_3-84a_2^2+15a^2a_1)x^{11}$$

(16)
$$-\frac{1}{12!}(9a_8-132aa_5-297a_1a_4-396a_2a_3+54a^2a_2+75aa_1^2)x^{12}$$

$$-\frac{1}{13!}(10a_9-217aa_6-564a_1a_5-855a_2a_4+159a^2a_3-480a_3^2 + 558aa_1a_2+120a_1^3-a^4)x^{13}$$

$$-\frac{1}{14!}(11a_{10}-338aa_7-1001a_1a_6-1716a_2a_5-2145a_3a_4+405a^2a_4 + 1728aa_1a_3+1353a_1^2a_2+1074aa_2^2-26a^3a_1)x^{14}$$

$$-\cdots,$$

$$y_3=\frac{1}{2}x^2-\frac{a}{5!}x^5-\frac{3a_1}{6!}x^6-\frac{3a_2}{7!}x^7-\frac{1}{8!}(10a_3-a^2)x^8$$

$$-\frac{1}{9!}(15a_4-9aa_1)x^9-\frac{1}{10!}(21a_5-27aa_2-21a_1^2)x^{10}$$

$$-\frac{1}{11!}(28a_6-66aa_3-132a_1a_2+a^3)x^{11}$$

$$-\frac{1}{12!}(36a_7 - 141aa_4 - 342a_1a_3 - 216a_2^2 + 18a^2a_1)x^{12}$$

$$-\frac{1}{13!}(45a_8 - 273aa_5 - 780a_1a_4 - 1170a_2a_3 + 72a_2a^2 + 111aa_1^2)x^{13}$$

$$-\frac{1}{14!}(55a_9 - 490aa_6 - 1617a_1a_5 - 2805a_2a_4 + 231a^2a_3$$
$$- 1650a_3^2 + 924aa_1a_2 + 231a_1^3 - a^4)x^{14}$$
$$-\cdots.$$

Let $Q(x_1, x_2, x_3) = 0$ be the equation of the osculating quartic. If we should substitute into its left member $x_\kappa = y_\kappa$ $(\kappa = 1, 2, 3)$, the coefficients of all powers of x up to and including the thirteenth would be equal to zero, and we should have fourteen equations for the ratios of the fifteen coefficients of Q. The equation $Q = 0$ has been obtained by the author in this way. The details of that solution, however, will not be given here, as the equation of the osculating quartic may be obtained more easily and in a more satisfactory form by another method. We wish, however, to emphasize the fact that the equations obtained by the two different methods have been compared, for the sake of checking the results presented here, and have been found to agree.

By combining y_1, y_2, y_3 so as to eliminate the terms in x up to the fourth and eighth orders, inclusive, Professor WILCZYNSKI has obtained as the equations of the osculating conic and cubic respectively:*

$$x_2^2 - 2x_1x_3 = 0, \tag{17}$$

$$7(15P_3P_3^{(3)} - 20P_3'P_3'' - 567P_3^3)\Omega_1(x) + 20[6P_3P_3'' - 7(P_3')^2]\Omega_2(x) = 0, \tag{18}$$

where

$$\Omega_1(x) = 5(x_2^2 - 2x_1x_3)(P_3'x_3 - 3P_3x_2) + 12P_3^2x_3^3, \tag{19}$$

$$\Omega_2(x) = 5(x_2^2 - 2x_1x_3)(21P_3x_1 - P_3''x_3) - 42P_3^2x_2x_3^2 - 14P_3P_3'x_3^3. \tag{20}$$

The curve $\Omega_1(x) = 0$ has a special significance for our problem. It is the only cubic of the pencil of cubics having eight consecutive points in common with C_y, which has a double point at P_{y_0}, and has therefore been called the eight-pointic nodal cubic.

The work will be simplified if we introduce a system of non-homogeneous coördinates X, Y defined by the following equations:

$$t_1 = y_1 - \frac{a_1}{3a}y_2 + \frac{a_1^2}{18a^2}y_3,$$

* *Proj. Diff. Geom.*, pp. 63, 64.

$$t_2 = \theta_1 y_2 - \frac{\theta_1 a_1}{3a} y_3, \tag{21}$$

$$t_3 = \theta_1^2 y_3,$$

$$X = \frac{t_2}{t_1}, \qquad Y = \frac{t_3}{t_1}, \tag{22}$$

in which θ_1 is an abbreviation for one of the cube roots of $-a/20$. The new triangle of reference, given by (21), is characterized by the fact that if Y be developed according to powers of X, the development assumes the form:

$$Y = \tfrac{1}{2}X^2 + X^5 + \alpha_7 X^7 + \cdots + \alpha_{14} X^{14} + \cdots, \tag{23}$$

in which all the coefficients are absolute invariants. Professor WILCZYNSKI has shown that in order to obtain the canonical form (23) of the development, the triangle of reference must be chosen as follows:* "One vertex is a point on the curve and one side of the triangle is the tangent at this point. The second side is the line upon which are situated the three points of inflection of the eight-pointic nodal cubic. The third side is the polar of the intersection of the other two with respect to the osculating conic. The numerical factors, which still remain arbitrary in a projective system of coördinates after the triangle of reference has been chosen, must be determined in such a way that the coördinates of one of the three points of inflection of the eight-pointic nodal cubic shall be $(0, -\sqrt[3]{16}, 1)$, and that the coördinates of the tangent to the cubic at this point shall be $(2\sqrt[3]{16}, 3\sqrt[3]{16^2}, 48)$." We are thus dealing with a coördinate system which has a purely projective relation to the curve C_y.

We now proceed to obtain the development (23) explicitly up to terms of the fourteenth order. In addition to the two fundamental invariants θ_3 and θ_8, it will be necessary to have explicit expressions for the following system of invariants of equation (10), obtained from θ_3 and θ_8 by the Jacobian process:

$$\begin{aligned} &\theta_{12} = 3\theta_3\theta_8' - 8\theta_8\theta_3', && \theta_{16} = \theta_3\theta_{12}' - 4\theta_{12}\theta_3', \\ &\theta_{20} = 3\theta_3\theta_{16}' - 16\theta_{16}\theta_3', && \theta_{24} = 3\theta_3\theta_{20}' - 20\theta_{20}\theta_3', \\ &\theta_{28} = \theta_3\theta_{24}' - 8\theta_{24}\theta_3', && \theta_{32} = 3\theta_3\theta_{28}' - 28\theta_{28}\theta_3', \end{aligned} \tag{24}$$

From (12), we find:

$$\theta_{12} = 2\cdot 3^2 a^2 a_3 - 2^3\cdot 3^2 a a_1 a_2 + 2^3\cdot 7 a_1^3,$$

$$\theta_{16} = 2\cdot 3^2 a^3 a_4 - 2^2\cdot 3^3 a^2 a_1 a_3 - 2^3\cdot 3^2 a^2 a_2^2 + 2^7\cdot 3 a a_1^2 a_2 - 2^5\cdot 7 a_1^4,$$

$$\theta_{20} = 2\cdot 3^3 a^4 a_5 - 2\cdot 3^2\cdot 5^2 a^3 a_1 a_4 + 2^3\cdot 3^2\cdot 31 a^2 a_1^2 a_3 - 2^2\cdot 3^3\cdot 7 a^3 a_2 a_3$$
$$+ 2^4\cdot 3^3\cdot 7 a^2 a_1 a_2^2 - 2^9\cdot 3\cdot 5 a a_1^3 a_2 + 2^9\cdot 7 a_1^5,$$

* *Proj. Diff. Geom.*, p. 86.

$$\theta_{24} = 2\cdot 3^4 a^5 a_6 - 2\cdot 3^4\cdot 11 a^4 a_1 a_5 - 2\cdot 3^3\cdot 67 a^4 a_2 a_4 + 2\cdot 3^2\cdot 647 a^3 a_1^2 a_4$$
$$- 2^2 3^4 7 a^4 a_3^2 + 2^2\cdot 3^5\cdot 41 a^3 a_1 a_2 a_3 - 2^4\cdot 3^2\cdot 13\cdot 29 a^2 a_1^3 a_3$$
$$- 2^5\cdot 3^4\cdot 43 a^2 a_1^2 a_2^2 + 2^{12}\cdot 3^2\cdot 5 a a_1^4 a_2 + 2^4\cdot 3^4\cdot 7 a^3 a_2^3 - 2^{11}\cdot 5\cdot 7 a_1^6,$$

$$\theta_{28} = 2\cdot 3^4 a^6 a_7 - 2^2\cdot 3^4\cdot 7 a^5 a_1 a_6 + 2\cdot 3^2\cdot 7\cdot 149 a^4 a_1^2 a_5 - 2^3\cdot 3^3\cdot 5^2 a^5 a_2 a_5$$
$$+ 2^4\cdot 3^2\cdot 7^2\cdot 11 a^4 a_1 a_2 a_4 - 2\cdot 3^3\cdot 151 a^5 a_3 a_4 - 2\cdot 3^2\cdot 7\cdot 893 a^3 a_1^3 a_4$$
(25)
$$+ 2^2\cdot 3^4\cdot 151 a^4 a_1 a_3^2 - 2^2\cdot 3^2\cdot 16251 a^3 a_1^2 a_2 a_3 + 2^2\cdot 3^6\cdot 23 a^4 a_2^2 a_3$$
$$+ 2^5\cdot 3^2\cdot 7\cdot 253 a^2 a_1^4 a_3 + 2^6\cdot 3^2\cdot 2441 a^2 a_1^3 a_2^2 - 2^4\cdot 3^6\cdot 23 a^3 a_1 a_2^3$$
$$- 2^{14}\cdot 3\cdot 5\cdot 7 a a_1^5 a_2 + 2^{14}\cdot 5\cdot 7 a_1^7,$$

$$\theta_{32} = 2\cdot 3^5 a^7 a_8 - 2^3\cdot 3^4\cdot 13 a^6 a_1 a_7 + 2\cdot 3^3\cdot 7\cdot 227 a^5 a_1^2 a_6 - 2^2\cdot 3^4\cdot 71 a^6 a_2 a_6$$
$$- 2\cdot 3^2\cdot 7\cdot 5063 a^4 a_1^3 a_5 + 2^2\cdot 3^4\cdot 1283 a^5 a_1 a_2 a_5 - 2\cdot 3^4\cdot 251 a^6 a_3 a_5$$
$$- 2\cdot 3^2\cdot 222757 a^4 a_1^2 a_2 a_4 + 2^2\cdot 3^3\cdot 4019 a^5 a_2 a_4 + 2\cdot 3^3\cdot 11711 a^5 a_1 a_3 a_4$$
$$- 2\cdot 3^4\cdot 151 a^6 a_4^2 + 2\cdot 3^2\cdot 7\cdot 29111 a^3 a_1^4 a_4 - 2^2\cdot 3^5\cdot 7\cdot 373 a^4 a_1^2 a_3^2$$
$$+ 2^2\cdot 3^5\cdot 5\cdot 113 a^5 a_2 a_3^2 + 2^2\cdot 3^3\cdot 237707 a^3 a_1^3 a_2 a_3$$
$$- 2^3\cdot 3^4\cdot 10799 a^4 a_1 a_2^2 a_3 - 2^6\cdot 3^2\cdot 7\cdot 4063 a^2 a_1^5 a_3 - 2^7\cdot 3^3\cdot 16417 a^2 a_1^4 a_2^2$$
$$+ 2^4\cdot 3^4\cdot 13697 a^3 a_1^2 a_2^3 - 2^4\cdot 3^7\cdot 23 a^4 a_2^4 + 2^{19}\cdot 3\cdot 5\cdot 7 a a_1^6 a_2 - 2^{16}\cdot 5\cdot 7^2 a_1^8.$$

We shall also need $\theta_{36} = 3\theta_3\theta'_{32} - 32\theta_{32}\theta'_3$. These Jacobians, together with θ_3 and θ_8, form a complete system (Σ) of invariants of equation (10), in the sense that any rational invariant whatever, involving $\theta_3 \equiv a$, and its derivatives, $\theta_3^{(i)} \equiv a_i\,(i = 1, 2 \cdots 9)$, may be expressed rationally in terms of the invariants of the system.* Equations (25) show that a, a_2, a_3, $\cdots$, a_9 may be expressed rationally in terms of the members of Σ and of a_1. Since the coefficients α_i in the development (23) are absolute invariants, when a and its derivatives have been replaced by the members of this system, the terms in α_i involving a_1 must in the aggregate disappear. We may therefore neglect, at the outset, all terms involving a_1; and it will be understood that terms involving a_1 have been omitted from the right hand members of all the equations which follow, up to and including (31). Equations (22) now take the simple form:

(26)
$$X = \frac{\theta_1 y_2}{y_1}, \qquad Y = \frac{\theta_1^2 y_3}{y_1};$$

or more explicitly:

$$\theta_1^{-1} X = x + \frac{3}{4!} a x^4 + \frac{3}{6!} a_2 x^6 + \frac{3}{7!}(33a^2 + a_3) x^7 + \frac{3}{8!} a_4 x^8$$

* *Proj. Diff. Geom.*, p. 36.

$$+\frac{3}{9!}(183aa_2+a_5)x^9+\frac{3}{10!}(a_6+273aa_3+3753a^3)x^{10}$$

$$+\frac{3}{11!}(a_7+388aa_4+413a_2^2)x^{11}+\frac{3}{12!}(a_8+531aa_5+1512a_2a_3$$

$$+60165a^2a_2)x^{12}+\frac{3}{13!}(a_9+705aa_6+2586a_2a_4$$

$$+118377a^2a_3+1512a_3^2+1017441a^4)x^{13}+\cdots,$$

(27)

$$\theta_1^{-2}Y=\frac{1}{2}x^2+\frac{9}{5!}ax^5+\frac{15}{7!}a_2x^7+\frac{9}{8!}(2a_3+53a^2)x^8+\frac{21}{9!}a_4x^9$$

$$+\frac{24}{10!}(a_5+150aa_2)x^{10}+\frac{9}{11!}(3a_6+671aa_3+8289a^3)x^{11}$$

$$+\frac{3}{12!}(10a_7+3171aa_4+3570a_2^2)x^{12}+\frac{3}{13!}(11a_8+4758aa_5$$

$$+14508a_2a_3+497772a^2a_2)x^{13}+\frac{3}{14!}(12a_9+6867aa_6$$

(28)

$$+27195a_2a_4+1069173a^2a_3+16020a_3^2+8580087a^4)x^{14}+\cdots.$$

Reverting the series (27), we have:

$$x=\theta_1^{-1}X-\frac{3}{4!}a\theta_1^{-4}X^4-\frac{3}{6!}a_2\theta_1^{-6}X^6+\frac{3}{7!}(72a^2-a_3)\theta_1^{-7}X^7$$

$$-\frac{3}{8!}a_4\theta_1^{-8}X^8+\frac{3}{9!}(447aa_2-a_5)\theta_1^{-9}X^9+\frac{3}{10!}(717aa_3-23058a^3$$

$$-a_6)\theta_1^{-10}X^{10}+\frac{3}{11!}(1097aa_4+973a_2^2-a_7)\theta_1^{-11}X^{11}+\frac{3}{12!}(1614aa_5$$

$$+3636a_2a_3-443691a^2a_2-a_8)\theta_1^{-12}X^{12}+\frac{3}{13!}(2298aa_6+6423a_2a_4$$

$$-985815a^2a_3+3636a_3^2+18311440a^4-a_9)\theta_1^{-13}X^{13}+\cdots.$$

(29)

Substituting this value of x in equation (28), we find the following symbolic expression for the development (23):

$$Y=\frac{1}{2}X^2+X^5-\frac{6}{7!}a_2\theta_1^{-5}X^7+\frac{6}{8!}(105a^2-a_3)\theta_1^{-6}X^8-\frac{6}{9!}a_4\theta_1^{-7}X^9$$

$$+\frac{6}{10!}(630aa_2-a_5)\theta_1^{-8}X^{10}+\frac{6}{11!}(990aa_3-43848a^3-a_6)\theta_1^{-9}X^{11}$$

$$(30)\qquad + \frac{6}{12!}(1485aa_4 + 1386a_2^2 - a_7)\theta_1^{-10}X^{12} + \frac{6}{13!}(2145aa_5 + 5148a_2a_3 - 806922a^2a_2 - a_8)\theta_1^{-11}X^{13} + \frac{6}{14!}(3003a_6 + 9009a_2a_4 - 1741284a^2a_3 + 5148a_3^2 + 46540494a^4 - a_9)\theta_1^{-12}X^{14} + \cdots.$$

It remains to replace the quantities a_i by the members of the system of invariants Σ of the differential equation (10). Solving equations (25) for these quantities, we find:

$$a = \theta_3, \qquad 2\cdot 3\theta_3\cdot a_2 = \theta_8, \qquad 2\cdot 3^2\theta_3^2\cdot a_3 = \theta_{12},$$

$$2\cdot 3^2\theta_3^3\cdot a_4 = \theta_{16} + 2\theta_8^2, \qquad 2\cdot 3^3\theta_3^4\cdot a_5 = \theta_{20} + 7\theta_{12}\theta_8,$$

$$2^2\cdot 3^4\theta_3^5\cdot a_6 = 2\theta_{24} + 67\theta_{16}\theta_8 + 2\cdot 7\theta_{12}^2 + 2\cdot 5^2\theta_8^3,$$

$$2^2\cdot 3^5\theta_3^6\cdot a_7 = 2\cdot 3\theta_{28} + 2^2\cdot 5^2\theta_{20}\theta_8 + 151\theta_{16}\theta_{12} + 3\cdot 127\theta_{12}\theta_8^2,$$

$$(31)\qquad 2^2\cdot 3^6\theta_3^7\cdot a_8 = 2\cdot 3\theta_{32} + 2\cdot 71\theta_{24}\theta_8 + 2\cdot 3\cdot 5^2\cdot 17\theta_{16}\theta_8^2 + 251\theta_{20}\theta_{12} + 3\cdot 151\theta_{16}^2 + 2^5\cdot 3\cdot 11\theta_{12}^2\theta_8 + 2\cdot 3\cdot 5^2\cdot 7\theta_8^4,$$

$$2^2\cdot 3^7\theta_3^8\cdot a_9 = 2\cdot 3\theta_{36} + 2\cdot 3\cdot 97\theta_{28}\theta_8 + 3\cdot 131\theta_{24}\theta_{12} + 3\cdot 7\cdot 79\theta_{20}\theta_{16} + 2\cdot 5^2\cdot 103\theta_{20}\theta_8^2 + 2\cdot 7681\theta_{16}\theta_{12}\theta_8 + 2^5\cdot 3\cdot 11\theta_{12}^3 + 2\cdot 3\cdot 2351\theta_{12}\theta_8^3,$$

where we still neglect to write the terms involving a_1. The required coefficients α_i are therefore the following absolute invariants:

$$\alpha_7 = 2^{-2}\cdot 3^{-2}\cdot 7^{-1}\theta_3^{-2}\theta_1^{-2}\theta_8,$$

$$\alpha_8 = -2^{-3}\cdot 3^{-3}\cdot 5\cdot 7^{-1}\theta_3^{-4}(\theta_{12} - 2\cdot 3^3\cdot 5\cdot 7\theta_3^4),$$

$$\alpha_9 = -2^{-3}\cdot 3^{-5}\cdot 5\cdot 7^{-1}\theta_3^{-5}\theta_1^{-1}(\theta_{16} + 2\theta_8^2),$$

$$\alpha_{10} = -2^{-4}\cdot 3^{-6}\cdot 7^{-1}\theta_3^{-6}\theta_1^{-2}(\theta_{20} + 7\theta_{12}\theta_8 - 2\cdot 3^4\cdot 5\cdot 7\theta_8\theta_3^4),$$

$$\alpha_{11} = 2^{-3}\cdot 3^{-7}\cdot 5\cdot 7^{-1}\cdot 11^{-1}\theta_3^{-8}(2\theta_{24} + 67\theta_{16}\theta_8 + 2\cdot 7\theta_{12}^2 - 2^2\cdot 3^4\cdot 5\cdot 11\theta_{12}\theta_3^4 + 2\cdot 5^2\theta_8^3 + 2^5\cdot 3^7\cdot 7\cdot 29\theta_3^8),$$

$$\alpha_{12} = 2^{-5}\cdot 3^{-9}\cdot 5\cdot 7^{-1}\cdot 11^{-1}\theta_3^{-9}\theta_1^{-1}(2\cdot 3\theta_{28} + 2^2\cdot 5^2\theta_{20}\theta_8 + 151\theta_{16}\theta_{12} + 3\cdot 127\theta_{12}\theta_8^2 - 2\cdot 3^6\cdot 5\cdot 11\theta_{16}\theta_3^4 - 2\cdot 3^5\cdot 11\cdot 37\theta_8^2\theta_3^4),$$

$$(32)\qquad \alpha_{13} = 2^{-5}\cdot 3^{-10}\cdot 5\cdot 7^{-1}\cdot 11^{-1}\cdot 13^{-1}\theta_3^{-10}\theta_1^{-2}(2\cdot 3\theta_{32} + 2\cdot 71\theta_{24}\theta_8 + 251\theta_{20}\theta_{12} - 2\cdot 3^4\cdot 5\cdot 11\cdot 13\theta_{20}\theta_3^4 + 3\cdot 151\theta_{16}^2 + 2\cdot 3\cdot 5^2\cdot 17\theta_{16}\theta_8^2 + 2^5\cdot 3\cdot 11\theta_{12}^2\theta_8 - 2\cdot 3^4\cdot 11\cdot 13\cdot 41\theta_{12}\theta_8\theta_3^4 + 2\cdot 3\cdot 5^2\cdot 7\theta_8^4 + 2^2\cdot 3^9\cdot 17\cdot 293\theta_8\theta_3^8),$$

$$\alpha_{14}=-2^{-4}\cdot 3^{-11}\cdot 5^{2}\cdot 7^{-2}\cdot 11^{-1}\cdot 13^{-1}\theta_3^{-12}(2\cdot 3\theta_{36}+2\cdot 3\cdot 97\theta_{28}\theta_8+3\cdot 131\theta_{24}\theta_{12}$$
$$-2\cdot 3^{4}\cdot 7\cdot 11\cdot 13\theta_{24}\theta_3^{4}+3\cdot 7\cdot 79\theta_{20}\theta_{16}+2\cdot 5^{2}\cdot 103\theta_{20}\theta_8^{2}$$
$$+2\cdot 7681\theta_{16}\theta_{12}\theta_8-2^{2}\cdot 3^{4}\cdot 7\cdot 11\cdot 13\cdot 19\theta_{16}\theta_8\theta_3^{4}+2^{5}\cdot 3\cdot 11\theta_{12}^{3}$$
$$-2\cdot 3^{4}\cdot 5\cdot 11^{2}\cdot 13\theta_{12}^{2}\theta_3^{4}+2^{3}\cdot 3^{3}\cdot 16123\theta_{12}\theta_3^{8}+2\cdot 3\cdot 2351\theta_{12}\theta_8^{3}$$
$$-2^{2}\cdot 3^{4}\cdot 7\cdot 11\cdot 13\cdot 17\theta_8^{3}\theta_3^{4}-2^{3}\cdot 3^{11}\cdot 7^{2}\cdot 11\cdot 13\cdot 41\theta_3^{12}).$$

Let the equation of the osculating quartic be assumed in the form:

$$(33)\qquad Q_Y=\delta_{3333}Y^4+\delta_{3332}Y^3X+\delta_{3322}Y^2X^2+\delta_{3222}YX^3+\delta_{2222}X^4+\delta_{333}Y^3+\delta_{332}Y^2X$$
$$+\delta_{322}YX^2+\delta_{222}X^3+\delta_{33}Y^2+\delta_{32}YX+\delta_{22}X^2+\delta_3Y+\delta_2X+\delta=0.$$

If we substitute the development (23) for Y into the left member of this equation, the coefficients of all powers of X up to and including the thirteenth must be equal to zero, that is, we must have:

$$\delta=\delta_2=0,\qquad \delta_{22}+\tfrac{1}{2}\delta_3=0,\qquad \delta_{222}+\tfrac{1}{2}\delta_{32}=0,\qquad \tfrac{1}{4}\delta_{33}+\tfrac{1}{2}\delta_{322}+\delta_{2222}=0,$$
$$\delta_{32}+\tfrac{1}{8}\delta_{333}+\tfrac{1}{4}\delta_{3322}=0,\qquad \delta_3+\tfrac{1}{4}\delta_{332}+\tfrac{1}{2}\delta_{3222}=0,$$
$$\alpha_7\delta_3+\delta_{33}+\delta_{322}+\tfrac{1}{8}\delta_{3332}=0,\qquad \alpha_8\delta_3+\alpha_7\delta_{32}+\delta_{332}+\delta_{3222}+\tfrac{1}{16}\delta_{3333}=0,$$
$$\alpha_9\delta_3+\alpha_8\delta_{32}+\alpha_7\delta_{33}+\alpha_7\delta_{322}+\tfrac{3}{4}\delta_{333}+\delta_{3322}=0,$$
$$(34)\qquad \alpha_{10}\delta_3+\alpha_9\delta_{32}+(1+\alpha_8)\delta_{33}+\alpha_8\delta_{322}+\alpha_7\delta_{332}+\alpha_7\delta_{3222}+\tfrac{3}{4}\delta_{3332}=0,$$
$$\alpha_{11}\delta_3+\alpha_{10}\delta_{32}+\alpha_9\delta_{33}+\alpha_9\delta_{322}+(1+\alpha_8)\delta_{332}+\alpha_8\delta_{3222}+\alpha_7\delta_{3322}+\tfrac{3}{4}\alpha_7\delta_{333}+\tfrac{1}{2}\delta_{3333}=0,$$
$$\alpha_{12}\delta_3+\alpha_{11}\delta_{32}+(\alpha_{10}+2\alpha_7)\delta_{33}+\alpha_{10}\delta_{322}+\alpha_9\delta_{332}+(\tfrac{3}{4}\alpha_8+\tfrac{3}{2})\delta_{333}$$
$$+\alpha_9\delta_{3222}+(1+\alpha_8)\delta_{3322}+\tfrac{3}{4}\alpha_7\delta_{3332}=0,$$
$$\alpha_{13}\delta_3+\alpha_{12}\delta_{32}+(\alpha_{11}+2\alpha_8)\delta_{33}+\alpha_{11}\delta_{322}+(\alpha_{10}+2\alpha_7)\delta_{332}+\tfrac{3}{4}\alpha_9\delta_{333}$$
$$+\alpha_{10}\delta_{3222}+\alpha_9\delta_{3322}+(\tfrac{3}{4}\alpha_8+\tfrac{3}{2})\delta_{3332}+\tfrac{3}{8}\alpha_7\delta_{3333}=0.$$

Solving these equations, we find:

$$\delta_{22}=-2\begin{vmatrix} d_{11} & d_{21} & d_{31}\\ d_{12} & d_{22} & d_{32}\\ d_{13} & d_{23} & d_{33}\end{vmatrix},\qquad \delta_{222}=2\begin{vmatrix} d_{11} & d_{21} & d_{41}\\ d_{12} & d_{22} & d_{42}\\ d_{13} & d_{23} & d_{43}\end{vmatrix},$$

$$(35)\qquad \delta_{33}-4\delta_{2222}=8\begin{vmatrix} d_{11} & d_{21} & d_{41}\\ d_{12} & d_{32} & d_{42}\\ d_{13} & d_{23} & d_{43}\end{vmatrix},\qquad \Delta_{2222}=2\begin{vmatrix} d_{21} & d_{31} & d_{41}\\ d_{22} & d_{32} & d_{42}\\ d_{23} & d_{33} & d_{43}\end{vmatrix},$$

$$\delta_{2222}=\begin{vmatrix} \alpha_7 & \alpha_8-4 & \alpha_9 & \alpha_{10} \\ d_{11} & d_{21} & d_{31} & d_{41} \\ d_{12} & d_{22} & d_{32} & d_{42} \\ d_{13} & d_{23} & d_{33} & d_{43} \end{vmatrix}, \qquad \delta_{332}=-4\Delta_{2222}-32\delta_{22},$$

where

$$\begin{aligned} &d_{11}=\alpha_8-6, && d_{31}=\alpha_{10}-\alpha_8\alpha_7-8\alpha_7, \\ &d_{12}=\alpha_9-2\alpha_7^2, && d_{32}=\alpha_{11}-2\alpha_9\alpha_7-\alpha_8^2-4\alpha_8+12, \\ &d_{13}=\alpha_{10}-2\alpha_8\alpha_7-4\alpha_7, && d_{33}=\alpha_{12}-3\alpha_8\alpha_9-8\alpha_7^2, \\ &d_{21}=\alpha_9-\alpha_7^2, && d_{41}=\alpha_{11}-\alpha_9\alpha_7-2\alpha_8-32, \\ &d_{22}=\alpha_{10}-3\alpha_8\alpha_7+2\alpha_7, && d_{42}=\alpha_{12}-2\alpha_{10}\alpha_7-\alpha_9\alpha_8+2\alpha_9-6\alpha_7^2, \\ &d_{23}=\alpha_{11}-\alpha_9\alpha_7-2\alpha_8^2+6\alpha_8-12, && d_{43}=\alpha_{13}+6\alpha_{10}-2\alpha_{10}\alpha_8-\alpha_9^2-14\alpha_8\alpha_7-28\alpha_7; \end{aligned} \tag{36}$$

so that the equation of the osculating quartic is

$$\begin{aligned} &(\delta_{33}-4\delta_{2222})(2Y-X^2-8XY^2)Y-\delta_{332}(X^3+16Y^3-2XY)Y \\ &+2\delta_{2222}(X^2-2Y)^2+2\delta_{222}(X^2-2Y)X+2\delta_{22}(X^2-2Y) \\ &+[2\alpha_7(\delta_{33}-4\delta_{2222})-8\alpha_9\delta_{22}-8(\alpha_8-4)\delta_{222}](X^2-2Y)Y^2 \\ &+64[(\alpha_8-2)\delta_{22}+\alpha_7\delta_{222}]Y^4+8(4\alpha_7\delta_{22}Y^2+2\delta_{222}XY+\delta_{22}X^2)XY=0, \end{aligned} \tag{37}$$

where all of the coefficients are absolute invariants. This is easily reduced to homogeneous form, if desired, by equations (22). Referred to the same triangle, the equations of the osculating conic and cubic are respectively:

$$X^2-2Y=0, \tag{38}$$

$$\begin{aligned} &\alpha_7(2Y-X^2-8XY^2)+2\alpha_7^2(X^2-2Y)Y \\ &\qquad +(\alpha_8-4)(X^3+16Y^3-2XY)=0. \end{aligned} \tag{39}$$

When $\alpha_7=0$, (39) reduces to the equation of the eight-pointic nodal cubic.

If we substitute the canonical development (23) of the curve C_y into the left member of the equation (33) of the osculating quartic, we find that the coefficient of X^{14} is

$$\begin{aligned} &\alpha_{14}\delta_3+\alpha_{13}\delta_{32}+(\alpha_{12}+2\alpha_9+\alpha_7^2)\delta_{33}+\alpha_{12}\delta_{322}+(\alpha_{11}+2\alpha_8)\delta_{332}+(\tfrac{3}{4}\alpha_{10}+3\alpha_7)\delta_{333} \\ &\qquad +\alpha_{11}\delta_{3222}+(\alpha_{10}+2\alpha_7)\delta_{3322}+\tfrac{3}{4}\alpha_9\delta_{3332}+(\tfrac{1}{2}\alpha_8+\tfrac{3}{2})\delta_{3333}. \end{aligned}$$

When this expression vanishes, the osculating quartic has fifteen consecutive

points in common with C_y at P_{y_0}, or, as we may say, hyperosculates C_y at that point. This condition may be put into the form:

$$K = \begin{vmatrix} d_{11} & d_{21} & d_{31} & d_{41} \\ d_{12} & d_{22} & d_{32} & d_{42} \\ d_{13} & d_{23} & d_{33} & d_{43} \\ d_{14} & d_{24} & d_{34} & d_{44} \end{vmatrix} = 0, \tag{40}$$

where

$$\begin{aligned} d_{14} &= \alpha_{11} - 2\alpha_9\alpha_7 - 4\alpha_8 - \alpha_7^3 - 24, \\ d_{24} &= \alpha_{12} - \alpha_{10}\alpha_7 - 2\alpha_9\alpha_8 + 6\alpha_9 - \alpha_8\alpha_7^2 - 2\alpha_7^2, \\ d_{34} &= \alpha_{13} - \alpha_{10}\alpha_8 - 2\alpha_9^2 - \alpha_9\alpha_7^2 - 16\alpha_8\alpha_7, \\ d_{44} &= \alpha_{14} + 6\alpha_{11} - 3\alpha_{10}\alpha_9 - 14\alpha_9\alpha_7 - \alpha_{10}\alpha_7^2 - 8\alpha_8^2 - 40\alpha_8 - 144. \end{aligned} \tag{41}$$

If the invariant equation (40) is satisfied for all values of x, that is, at all points of the curve C_y, this curve is itself a quartic.

UNIVERSITY OF ILLINOIS,
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NOTE ON A SYSTEM OF AXIOMS FOR GEOMETRY*

BY

ARTHUR RICHARD SCHWEITZER

§1.

In these Transactions, vol. 3 (1902), p. 142, E. H. MOORE has given a set of projective axioms for geometry. In further development of the standpoint of MOORE a system of axioms for three dimensional Euclidian geometry has been constructed by VEBLEN† by means of the "betweenness" relation.‡

A feature of VEBLEN's system is that a planar axiom (axiom VIII, l. c.) is necessary to establish the existence of an infinitude of points and to prove the theorem, "To any four distinct points of a line the notation A, B, C, D may always be assigned so that they are in the order $ABCD$." In this note I give a set of independent axioms in terms of sameness of sense of dyads, abstractly expressed by $\alpha\beta K\gamma\delta$§, which implies VEBLEN's axioms I–VIII, but which does not require a planar axiom for the proof of the preceding properties. Of the two relations, sameness of sense and betweenness, one is definable in terms of the other; but the two definitions are not equally simple, as will appear below.

§2.

The axioms and definitions in terms of the relation K are as follows:‖

I. *Axioms.*

1. There exist at least two distinct points.¶
2. $\alpha\beta K$** implies $\alpha\beta\overline{K}\beta\alpha$.††

* Part of a paper presented to the Society (Chicago) April 14, 1906.

† These Transactions, vol. 5 (1904), p. 343.

‡ VEBLEN expresses that B is between A, C by saying that A, B, C are in the *order* ABC; thus the triple ABC is an undefined symbol. In my theory of geometrical relations (which will appear elsewhere in extended form) the order ABC is expressed by the relational statement $(ABC)R(ABC)$; i. e., A, B, C are mutually related.

§ The K is a symbol of an undefined relation which may be read, "the order $\alpha\beta$ is identical with the order $\gamma\delta$."

‖ In the phrasing of the axioms I have found the systems of MOORE and VEBLEN instructive. Instead of the relation K between dyads a class of VAILATI relations between points may be used; I show this in detail elsewhere.

¶ Greek letters are used to denote points.

** That is, $\alpha\beta K\alpha\beta$.

†† The rule over the K is a symbol of negation.

3. $\alpha\beta K\gamma\delta$ implies $\beta\alpha K\delta\gamma$.

4. $\alpha \neq \beta$ implies the existence of ξ such that either ($\alpha\xi K\alpha\beta$ and $\alpha\xi K\beta\xi$) or ($\xi\alpha K\beta\alpha$ and $\xi\alpha K\xi\beta$).

5. $\alpha\beta K\gamma\delta$ implies $\gamma\delta K\alpha\beta$.

5′. $\alpha\beta K\xi\eta$, $\xi\eta K\gamma\delta$ imply $\alpha\beta K\gamma\delta$.

6. $\alpha\beta K$, $\beta\xi K$, $\alpha\beta K\beta\xi$ imply $\alpha\beta K\alpha\xi$.

6′. $\alpha\beta K\gamma\delta$, $\alpha \neq \gamma$, $\beta \neq \delta$ imply $\alpha\beta K\alpha\gamma$ or $\alpha\beta K\gamma\alpha$.

7. $\alpha\beta K$, $\beta\gamma K$, $\gamma\alpha K$, $\alpha \neq \beta$, $\beta \neq \gamma$, $\alpha \neq \gamma$ imply the existence of ξ, η, ζ such that $\xi\eta K$, $\eta\zeta K$, $\zeta\xi K$ and $\xi\eta \overline{K} \xi\zeta$, $\xi\eta \overline{K} \zeta\eta$.

8. $\alpha\beta K$, $\beta\gamma K$, $\gamma\alpha K$, $\alpha\beta \overline{K} \beta\gamma$, $\alpha\beta \overline{K} \gamma\beta$, $\alpha\beta K\beta\epsilon$, $\beta\delta K\delta\gamma$ imply the existence of ξ such that $\alpha\xi K\xi\gamma$ and ($\delta\epsilon K\xi\epsilon$ or $\delta\epsilon K\delta\xi$) or ($\epsilon\xi K\delta\xi$ or $\epsilon\xi K\epsilon\delta$) or ($\xi\delta K\epsilon\delta$ or $\xi\delta K\xi\epsilon$).*

II. *Definitions.*

1. α is *between* $\beta\gamma$ means, $\beta\gamma K\alpha\gamma$ and $\beta\gamma K\beta\alpha$.† That is, $\beta\alpha K\alpha\gamma K\beta\gamma$, where the order of the dyadic terms, under the axioms, is immaterial.

2. α is *collinear* with $\beta\gamma$ means, $\beta\gamma K\alpha\gamma$ or $\beta\gamma K\beta\alpha$.

Thus, three points α, β, γ form a triangle if, and only if, $\alpha\beta K$, $\beta\gamma K$, $\gamma\alpha K$ and $\beta\gamma \overline{K} \alpha\gamma$, $\beta\gamma \overline{K} \beta\alpha$.

§ 3.

By means of the preceding axioms, the following theorems may be proved.‡

(1) $\alpha \neq \beta$ implies $\alpha\beta K$.

By 4 we have $\alpha\xi K\alpha\beta$; by 5, 5′, $\alpha\beta K$.

(2) $\alpha\beta K\gamma\delta$ implies $\alpha\beta \overline{K} \delta\gamma$, $\beta\alpha \overline{K} \gamma\delta$.

By 5, 5′, $\alpha\beta K\gamma\delta$ implies $\alpha\beta K$, $\gamma\delta K$ and hence by 2, $\alpha\beta \overline{K} \beta\alpha$, $\gamma\delta \overline{K} \delta\gamma$. Now if $\alpha\beta K\delta\gamma$, since $\alpha\beta K\gamma\delta$ we have by 5 and 5′ $\gamma\delta K\delta\gamma$, which contradicts axiom 2. Similarly we prove $\beta\alpha \overline{K} \gamma\delta$.

(3) $\alpha\beta K\gamma\delta$ implies $\alpha \neq \beta$ and $\gamma \neq \delta$.

By theorem (2).

(4) If α is between $\beta\gamma$ then α is between $\gamma\beta$ and β is not between $\alpha\gamma$; also α, β, γ are distinct.

By definition 1 and the hypothesis we have $\beta\gamma K\alpha\gamma$ and $\beta\gamma K\beta\alpha$; hence by 3, $\gamma\beta K\gamma\alpha$ and $\gamma\beta K\alpha\beta$.

If β is between $\alpha\gamma$ we have $\alpha\gamma K\alpha\beta$ and $\alpha\gamma K\beta\gamma$; hence by 5, 5′ $\beta\gamma K\alpha\beta$, which contradicts $\beta\gamma K\beta\alpha$ by (2).

From $\gamma\beta K\gamma\alpha$ and $\gamma\beta K\alpha\beta$, by (3) it follows that α, β, γ are distinct.

(5) If $\alpha \neq \beta$ then there is a point ξ' such that β is between $\alpha\xi'$, and a point ξ'' such that α is between $\beta\xi''$.

* Under the axioms the terms of the major disjunctions are equivalent.

† Compare with this definition GRASSMANN, *Gesammelte Werke*, vol. I, p. 7.

‡ In the demonstrations, numbers without parentheses refer to axioms of § 2; numbers enclosed in parentheses refer to theorems.

By 4, if $\alpha \neq \beta$ there exists ξ' such that β is between $\alpha\xi'$; interchanging α and β, we find by 4 a point ξ'' such that α is between $\beta\xi''$.

(6′) $\alpha\beta K\gamma\beta$, $\gamma \neq \alpha$ imply $\alpha\beta K\gamma\alpha$ or $\alpha\beta K\alpha\gamma$.

Since by (3) $\gamma \neq \beta$, there exists by 4, 5, 5′ a point ξ such that $\gamma\beta K\beta\xi$, i. e., by 6, $\gamma\beta K\gamma\xi$. From $\alpha\beta K\gamma\beta$ and $\gamma\beta K\gamma\xi$ it follows by 5′ that $\alpha\beta K\gamma\xi$. Hence by 6′, since $\gamma \neq \alpha$ and $\xi \neq \beta$, $\alpha\beta K\alpha\gamma$ or $\alpha\beta K\gamma\alpha$.

(6) If ξ is collinear with $\beta\gamma$ and $\xi \neq \beta$, γ, then ξ is between $\beta\gamma$ or β is between $\xi\gamma$ or γ is between $\xi\beta$, and conversely.

By definition and the hypothesis, $\beta\gamma K\xi\gamma$ or $\beta\gamma K\beta\xi$. If $\beta\gamma K\xi\gamma$, by (6′) we have $\beta\gamma K\beta\xi$ or $\beta\gamma K\xi\beta$. If $\beta\gamma K\xi\beta$, by 5, 5′, 6, $\beta\gamma K\xi\beta K\xi\gamma$ and hence β is between $\xi\gamma$. If $\beta\gamma K\beta\xi$, by (6′), 3, $\beta\gamma K\xi\gamma$ or $\beta\gamma K\gamma\xi$. If $\beta\gamma K\beta\xi$ and $\beta\gamma K\xi\gamma$ then ξ is between $\beta\gamma$; if $\beta\gamma K\beta\xi$ and $\beta\gamma K\gamma\xi$ then γ is between $\xi\beta$.

Conversely, if ξ is between $\beta\gamma$ then $\beta\gamma K\beta\xi$; if β is between $\xi\gamma$, $\beta\gamma K\xi\gamma$; if γ is between $\xi\beta$, $\beta\gamma K\beta\xi$.

(7) If $\gamma \neq \delta$ are collinear with $\alpha\beta$ then α is collinear with $\gamma\delta$.

By the hypothesis and definition 2, $\alpha\beta K\alpha\gamma$ or $\alpha\beta K\gamma\beta$ and $\alpha\beta K\alpha\delta$ or $\alpha\beta K\delta\beta$. To prove, $\gamma\delta K\gamma\alpha$ or $\gamma\delta K\alpha\delta$. We distinguish four cases, labelled I, II, III, IV respectively.

I. $\alpha\beta K\alpha\gamma$, $\alpha\beta K\delta\beta$.

By (6′) $\alpha\beta K\delta\beta$ implies $\alpha\beta K\delta\alpha$ or $\alpha\beta K\alpha\delta$. $\alpha\beta K\delta\alpha$ and $\alpha\beta K\alpha\gamma$ imply by 5, 5′, 6, 3, $\gamma\delta K\alpha\delta$. $\alpha\beta K\alpha\delta$ and $\alpha\beta K\alpha\gamma$ imply by 5, 5′, $\alpha\delta K\alpha\gamma$. By (6′) $\alpha\delta K\alpha\gamma$ implies $\alpha\delta K\gamma\delta$ or $\alpha\delta K\delta\gamma$; if $\alpha\delta K\delta\gamma$, by 6, $\alpha\delta K\alpha\gamma$ and hence $\gamma\delta K\gamma\alpha$.

II. $\alpha\beta K\alpha\gamma$, $\alpha\beta K\alpha\delta$.

Hence $\alpha\gamma K\alpha\delta$; therefore by (6′) $\alpha\gamma K\gamma\delta$ or $\alpha\gamma K\delta\gamma$. If $\alpha\gamma K\gamma\delta$, then $\gamma\delta K\alpha\delta$. If $\alpha\gamma K\delta\gamma$ then $\gamma\delta K\gamma\alpha$.

III. $\alpha\beta K\gamma\beta$, $\alpha\beta K\alpha\delta$.

$\alpha\beta K\gamma\beta$ implies $\alpha\beta K\gamma\alpha$ or $\alpha\beta K\alpha\gamma$. If $\alpha\beta K\alpha\gamma$ and $\alpha\beta K\alpha\delta$ then we have II. If $\alpha\beta K\alpha\delta$ and $\alpha\beta K\gamma\alpha$ then $\gamma\alpha K\alpha\delta$, that is, $\gamma\delta K\gamma\alpha$.

IV. $\alpha\beta K\gamma\beta$, $\alpha\beta K\delta\beta$.

$\alpha\beta K\gamma\beta$ implies $\alpha\beta K\gamma\alpha$ or $\alpha\beta K\alpha\gamma$. For $\alpha\beta K\alpha\gamma$ and $\alpha\beta K\delta\beta$ see I. If $\alpha\beta K\gamma\alpha$ then $\gamma\alpha K\gamma\beta$, $\alpha\beta K\gamma\beta$. Also $\alpha\beta K\gamma\beta$, $\alpha\beta K\delta\beta$ imply $\gamma\beta K\delta\beta$; hence $\gamma\beta K\gamma\delta$ or $\gamma\beta K\delta\gamma$. If $\gamma\beta K\gamma\delta$, since $\gamma\alpha K\gamma\beta$, $\gamma\alpha K\gamma\delta$ and hence $\gamma\delta K\gamma\alpha$. Similarly, if $\gamma\beta K\delta\gamma$ then $\gamma\delta K\alpha\delta$.

(8) There exists an infinitude of points.

By axiom 1 and the repeated application of 4, using 5, 5′ we find points $\xi_1, \xi_2, \cdots, \xi_n$ such that

$$\alpha\beta K\beta\xi_1 K\xi_1\xi_2 K \cdots K\xi_{n-1}\xi_n.$$

By axioms 6 and 2 the points $\xi_1, \xi_2, \cdots, \xi_n$ are distinct. This proof, of course involves the principle of induction.

The proof of the order theorem mentioned in § 1 is easily obtained by prov-

ing VEBLEN's six lemmas, l. c., pp. 356–357 with the aid of 1–6′ only;* a direct proof is also readily given.

§4.

The independence systems for axioms 1–8 are very simple. We denote by S_i the set of elements such that in this set axiom i ($i = 1, 2, 3, \cdots, 8$) is contradicted and the remaining axioms are satisfied or non-effective.†

S_1.

One point, 1, such that $11\overline{K}$. Axiom 1 is contradicted; the remaining axioms are non-effective.

S_2.

Two points 1, 2, such that any two dyads (identical or not) of the following set are in the K relation: 11, 22, 12, 21. Axiom 1 is satisfied; axiom 2 is contradicted; axiom 3 is satisfied. Axiom 4 asserts $1 \neq 2$ implies the existence of ξ such that $1\xi K12$ and $1\xi K2\xi$ or $\xi 1K21$ and $\xi 1K\xi 2$; thus we may put $\xi = 1$ or 2 and hence axiom 4 is satisfied. Axioms 5, 5′, 6, 6′ are easily verified. Axioms 7 and 8 are non-effective.

S_3.

A VAILATI line:‡ $\alpha\beta K\gamma\delta$ if, and only if, α precedes β and γ precedes δ; $\alpha\beta K$ if, and only if, α precedes β. Axioms 1, 2 are satisfied; axiom 3 is contradicted. Axiom 4 is satisfied: the terms of the disjunction are mutually exclusive. Axioms 5, 5′, 6, 6′ are satisfied. Axioms 7, 8 are non-effective, since $\alpha\beta K$, $\beta\gamma K$ imply $\alpha\gamma K$.

S_4.

Two points, 1, 2, such that 12K, 21K, $12\overline{K}21$, $21\overline{K}12$. Axioms 1–3 are satisfied. Axiom 4 is contradicted. Axioms 5, 5′ are satisfied. Axiom 6 is non-effective, since $12\overline{K}21$, $21\overline{K}12$. Axiom 6′ is non-effective since only 12K, 21K. Since there are only two points axioms 7, 8 are non-effective.

S_5.

Two points, 1, 2, such that 11K12, 11K21, $12\overline{K}11$, $21\overline{K}11$, 22K21, 22K12, $21\overline{K}22$, $12\overline{K}22$, $11\overline{K}$, $12\overline{K}$, $22\overline{K}$, $21\overline{K}$, $11\overline{K}22$, $22\overline{K}11$. Axiom 1 is satisfied. Axiom 2 is non-effective. Axiom 3 is satisfied. Axiom 4 is satisfied since $1 \neq 2$ implies 1, 2 such that 11K12 and 11K21 and 22K21 and 22K12. Axiom 5 is contradicted. Axioms 5′–8 are non-effective.

$S_{5'}$.

Two points 1, 2 such that 11K12, 12K11, 22K21, 21K22, 11K21, 21K11, 22K12, 12K22, $11\overline{K}$, $12\overline{K}$, $22\overline{K}$, $21\overline{K}$, $11\overline{K}22$, $22\overline{K}11$. Axiom 1 is satisfied.

* As in the case of the preceding theorems, axiom 6′ is used here only through theorem (6′).

† The notation S_i is due to DICKSON, these Transactions, vol. 4 (1903), p. 16.

‡ *Rivista di Matematica* II, pp. 71–75.

Axiom 2 is non-effective. Axiom 3 is satisfied. Axiom 4 requires that $1 \neq 2$ implies the existence of ξ such that $1\xi K12$ and $1\xi K2\xi$ or $\xi 1K21$ and $\xi 1K\xi 2$; hence we may take $\xi = 1$. Similarly, we must have ξ such that $2\xi K21$ and $2\xi K1\xi$ or $\xi 2K12$ and $\xi 2K\xi 1$; hence we may take $\xi = 2$. Axiom 5 is satisfied, axiom 5′ is contradicted. Axioms 6–8 are non-effective.

S_6.

Seven points, 1–7, arranged in the following two sets of 21 dyads each:

I	II
12, 23, 34, 45, 56, 67, 71	21, 32, 43, 54, 65, 76, 17
13, 24, 35, 46, 57, 61, 72	31, 42, 53, 64, 75, 16, 27
41, 52, 63, 74, 15, 26, 37	14, 25, 36, 47, 51, 62, 73

$\alpha\beta K\gamma\delta$ if, and only if, $\alpha\beta$ and $\gamma\delta$ are both of set I or both of set II. $\alpha\beta K$ if, and only if, $\alpha\beta$ is of set I or of set II. Axioms 1–5′ are satisfied. Axiom 6 is contradicted. Axioms 6′–7 are satisfied; axiom 8 is non-effective.

$S_{6'}$.

A cartesian plane having rational coördinates; $\alpha\beta K\gamma\delta$ if, and only if, $\alpha\beta$ and $\gamma\delta$ are two parallel segments similarly directed. Axiom 6′ is contradicted. In verifying the remaining axioms we note that two parallel segments are collinear if they pass through the same point, and that any segment is parallel to itself. We observe that the property expressed by theorem (6′) under § 3 is verified.*

S_7.

A cartesian line having integral coördinates; $\alpha\beta K\gamma\delta$ if, and only if, the segment $\alpha\beta$ has the same sign as the segment $\gamma\delta$.

S_8.

A cartesian plane having integral coördinates.

§ 5.

To define $\alpha\beta K\gamma\delta$ in terms of VEBLEN's order, we find it desirable to introduce a point. We define as follows:

$\alpha\beta K\gamma\delta$ means, 1°. There exists a ξ such that ($\alpha\beta\xi$ and $\gamma\delta\xi$) or ($\beta\alpha\xi$ and $\delta\gamma\xi$).

2°. The existence of ξ such that $\alpha\beta\xi$ and $\gamma\delta\xi$ implies $\alpha\gamma\xi$ or $\gamma\beta\xi$; and the existence of ξ such that $\beta\alpha\xi$ and $\delta\gamma\xi$ implies $\beta\delta\xi$ or $\delta\alpha\xi$.

*This remark is important if we wish to extend axioms 1–8 to a set of axioms for Euclidian geometry (cf. VEBLEN, loc. cit.). Such extension may easily be made either by retaining axiom 6′ as stated, or by substituting for the latter an axiom with the content of theorem (6′). In each instance the parallel axiom will have a corresponding phrasing, and the relation K may be interpreted respectively by sameness of sense of (collinear) segments, similarity of direction of (parallel) segments.

In virtue of this definition it might seem that we could establish a correspondence between "sameness of sense" properties and "order" properties. This view seems, however, largely unprofitable; for example, that three points are collinear can be expressed much more concisely in terms of order than is inferred by this correspondence from the definition in terms of the relation K.† Thus it does not seem possible to convert definitionally one system of axioms into the other and to retain that elegance which is peculiar to each individual standpoint.

† VEBLEN, l. c.

IRREDUCIBLE HOMOGENEOUS LINEAR GROUPS IN AN ARBITRARY DOMAIN*

BY

WILLIAM BENJAMIN FITE

I have given elsewhere a necessary and sufficient condition that certain categories of abstract groups of finite order be simply isomorphic with irreducible homogeneous linear groups in the domain of all real and complex numbers.† In the present paper I establish a two-fold generalization of this result by showing that the same condition applies to all groups of finite order and to an arbitrary domain.

The proof for the necessity of this condition rests upon a conclusion drawn from Theorem II of SCHUR's *Neue Begründung der Theorie der Gruppencharaktere.*‡

Suppose that we have a homogeneous linear group G of finite order whose coefficients belong to Ω (an arbitrary finite field or an arbitrary domain) and which is irreducible in Ω. If P is a given substitution on the same f variables as G with coefficients in Ω, the equation

$$|P - xE_f| = 0$$

is an equation in the unknown x with known coefficients. If this equation has a root in Ω and if P is commutative with every substitution of G, SCHUR's proof of his Theorem II applies, therefore P must be a similarity substitution.

If Ω is any finite field, say the $GF[p^m]$, then in the field F_p, which is the aggregate of all the fields § $GF[p^i]$ $(i = 1, 2, \cdots)$, G can be so transformed that in the matrix of every substitution of its central H all the elements below the main diagonal are zero. If Ω is a domain, the same result can be accom-

* Presented to the Society, in a somewhat different form, December 31, 1908.

† Transactions of the American Mathematical Society, vol. 7 (1906), p. 65; Bulletin of the American Mathematical Society, vol. 14 (1908), p. 327.

‡ Berliner Sitzungsberichte, 1905, I, p. 410.

§ Cf. DICKSON, Transactions of the American Mathematical Society, vol. 8 (1907), p. 389; Bulletin of the American Mathematical Society, vol. 13 (1907), p. 477.

plished by passing to the domain of all real and complex numbers.* In this transformed form of G no two substitutions S_1 and S_2 of H can change x_j into the same function, since, if this were possible, $S_1 S_2^{-1}$ would leave x_j unchanged —that is, one of the multipliers of $S_1 S_2^{-1}$ would be unity. But, since this substitution is invariant in the irreducible group G, our conclusion from SCHUR's Theorem shows that this is possible only in case $S_1 S_2^{-1}$ is a similarity substitution with its multipliers all equal to unity—that is, in case $S_1 S_2^{-1}$ is the identical substitution. But S_1 and S_2 were taken as distinct substitutions.

If now H is of order $p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, where $p_1, p_2, \cdots, p_r$, are distinct primes, it is the direct product of groups $H_1, H_2, \cdots, H_r$, of order $p_1^{a_1}, p_2^{a_2}, \cdots, p_r^{a_r}$, respectively; and it follows from what has just been proved that $H_i (i=1, 2, \cdots, r)$, can contain only one subgroup of order p_i and is therefore cyclic.† *Hence H is cyclic.*

Suppose now that G is any group of finite order g with a cyclic central generated by the operation h of order $a\ (>1)$. If we write G as a regular permutation group h will take the form

$$h \equiv (x_{11} x_{12} \cdots x_{1a})(x_{21} x_{22} \cdots x_{2a}) \cdots (x_{\gamma 1} x_{\gamma 2} \cdots x_{\gamma a}),$$

where $\gamma = g/a$. If now we write this permutation group as a homogeneous linear group and then transform it by the substitution

$$S: \qquad y_{ij} = \sum_{k=1}^{a} \omega^{(k-1)(j-1)} x_{ik},$$

where ω is a primitive ath root of unity, h will be transformed into its normal form. Moreover in this transformed form of G the variables

$$x_{1j}, \ x_{2j}, \ \cdots, \ x_{\gamma j} \qquad (j=1, 2, \cdots, a)$$

are transformed among themselves in semi-canonical form.‡ We shall denote the group formed by the substitutions on these variables by G_j. If $j-1$ is relatively prime to a, G_j is simply isomorphic with G.

If G were not simply isomorphic with any irreducible group in C (the domain of all real and complex numbers), then when G_j ($j-1$ relatively prime to a) is completely reduced in C there must be some irreducible component such that when all the irreducible representations of G that are equivalent to this

* Cf. WEBER, *Algebra*, II (second edition), pp. 172, 173; 176–178. This discussion of WEBER's is with reference to the domain of all real and complex numbers, but the processes described are valid in the field F_p.

† BURNSIDE, *Theory of groups of finite order*, pp. 73, 75. If $p_i = 2$, H_i could not be the non-cyclic group with only one subgroup of order 2, since this contains more than one cyclic subgroup of order 4.

‡ Cf. Transactions of the American Mathematical Society, vol. 8 (1907), p. 108.

one (in number equal to the degree of each one*) are taken together, there is an r_1, r_2 isomorphism between the group formed by these representations and the group in the remaining variables, and $r_1 > 1$, $r_2 > 1$; since otherwise G would be simply isomorphic with some irreducible group. Call these two components L_1 and L_2 respectively and let their orders be denoted by l_1 and l_2 respectively. If H_i $(i = 1, 2)$ is the subgroup of L_i that corresponds to identity in the other component, H_i can contain no invariant substitution of L_i, except identity, since every invariant substitution G_j is a similarity substitution. Now each irreducible component of L_1 is of degree at most $\sqrt{l_1/a}$, since each such component is of order l_1 and contains at least a invariant substitutions.† But by virtue of the manner in which G_j is formed from L_1 and L_2 we must have $g = r_1 r_2 \cdot l_1/r_1$ and $l_1 = g/r_2$. Then the number of variables in L_1 is at most $g/(ar_2)$.

Any substitution of H_1, such as t of order b, when combined with identity of H_2 gives a substitution of G_j that has at least $g/a - g/(ar_2)$ multipliers equal to unity. Now there are $\phi(a)$ values of j such that $j - 1$ is relatively prime to a and there is a similar substitution in each G_j corresponding to these values of j. These similar substitutions combine with one another and with a certain substitution from every G_j ($j - 1$ not relatively prime to a) to form a certain substitution of G of order b.

Since every G_j is in semi-canonical form, in any substitution of G_j the variables are permuted among themselves (except for certain coefficients). If now $j - 1$ is not relatively prime to a, the cycles of the permutation in a substitution of G_j that corresponds to a non-invariant substitution of G must all be of the same order, since otherwise a certain power of this substitution that is not invariant in G would change the variables of at least one of the sets of variables into themselves multiplied by certain constants. This means that in the regular permutation group there would be a non-invariant operation that leaves some of the systems of intransitivity of H unchanged. But this is impossible in a regular group. Hence in G_j ($j - 1$ not relatively prime to a) the variables of t_j (the substitution that corresponds to t) are permuted among themselves (except for certain coefficients) in cycles of order b. The characteristic determinant D_j of t_j is then the product of g/ab determinants each of order b:

$$D_j \equiv \{(-1)^b \lambda^b - (-1)^b\}^{g/ab}.$$

Hence t_j has g/ab multipliers equal to unity. The total number of multipliers that equal unity in the substitution of G that corresponds to t would then be at least

$$\left(\frac{g}{a} - \frac{g}{ar_2}\right)\phi(a) + \frac{g}{ab}[a - \phi(a)].$$

* BURNSIDE, Acta Mathematica, vol. 28 (1904), p. 383.

† Transactions of the American Mathematical Society, vol. 7 (1906), p. 67.

But this number is greater than the actual number of multipliers of this substitution that equal unity, namely* g/b, provided $r_2 > 2$. However, if $r_2 = 2$, H_2 would contain only one substitution besides identity. This substitution would be invariant in L_2 and would combine with identity of L_1 to form an invariant substitution of G_j that is not a similarity substitution.

Hence G must be simply isomorphic with an irreducible group in C.

If G has no invariant operation except identity, we can form the direct product $\overline{G}$ of G and an operation h of order $a(>1)$. It follows from the preceding discussion that $\overline{G}$ is simply isomorphic with an irreducible group in C. The subgroup of this irreducible group that corresponds to G must also be irreducible, since otherwise, in view of the fact that the substitution corresponding to h is a similarity substitution, the group that is simply isomorphic with $\overline{G}$ would be reducible.

Hence any group of finite order with a cyclic central is simply isomorphic with a homogeneous linear group that is irreducible in C. But every irreducible representation of G (in C) is equivalent to one of the irreducible components of the regular permutation group that is simply isomorphic with G when this is written as a homogeneous linear group.† If now no irreducible component of this latter group in any domain Ω were simply isomorphic with G, then no irreducible component in C could be simply isomorphic with G.

We can formulate the results of the preceding discussion into the

THEOREM. *A necessary and sufficient condition that any group of finite order be simply isomorphic with an irreducible group in any domain is that its central be cyclic.*

It should be remarked that the sufficiency of this condition is established only for an arbitrary domain, whereas the necessity of it is established also for any finite field.

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*Transactions of the American Mathematical Society, vol. 8 (1907), p. 108.

† BURNSIDE, loc. cit., p. 387.

ON THE INTEGRATION OF THE HOMOGENEOUS LINEAR DIFFERENCE EQUATION OF SECOND ORDER*

BY

WALTER B. FORD.

I. *Introduction.*

1. It is proposed in the present paper to study the behavior for large positive integral values of x of the general solution of the equation

$$a_0(x)u(x+2)+a_1(x)u(x+1)+a_2(x)u(x)=0, \tag{1}$$

the coefficients $a_0(x)$, $a_1(x)$, $a_2(x)$ being given functions (real or complex) of x and subject only to conditions relative to the form taken by the expression

$$4\frac{a_0(x)a_2(x+1)}{a_1(x)a_1(x+1)}-1$$

when x is very large.† The general results are summarized in Theorem I, after which application is made to the following special equation considered by HORN: ‡

$$P_0(x)u(x+2)+x^kP_1(x)u(x+1)+x^{2k}P_2(x)u(x)=0,$$

where k is any integer, positive, negative or zero and $P_0(x)$, $P_1(x)$, $P_2(x)$ are developable asymptotically (or as convergent series) in the form

$$P_\lambda(x)=a_\lambda+\frac{b_\lambda}{x}+\frac{c_\lambda}{x^2}+\cdots+\frac{p_\lambda+\omega_\lambda(x)}{x^n}\qquad(\lim_{x=\infty}\omega_\lambda(x)=0).$$

The results obtained for this type of equation are stated in Theorem II and are shown to be in accord with those obtained by HORN. Also certain results addi-

* Presented to the Society (Chicago) April 17, 1908.

† Papers relating to this subject but placing more restrictive conditions upon the coefficients have been published by HORN, Mathematische Annalen, vol. 53 (1900), pp. 117-192, and by the present author, Annali di Matematica, series 3, vol. 13 (1907), pp. 313-328. For a summary of literature concerning equations of form (1) see BARNES, Messenger of Mathematics, vol. 34 (1904), p. 53.

‡ *Loc. cit.*, p. 190. For simplicity we shall here confine ourselves to equations of order 2, whereas HORN considers similar equations of any order. A generalization of our results to any order is immediate, in view of the general character of the investigations contained in the above mentioned memoir in the Annali upon which the present paper depends.

tional to his are to be here found, and application of these is made in § 11 to the study of the behavior of Legendre's function of the first kind P_n for large values of n.

II. *Reduction of the Equation.*

2. We shall first show that in case $a_1(x) \neq 0$, $a_2(x) \neq 0$ for all $x \geqq \alpha = constant$,* the study of equation (1) may be made to depend upon that of a certain linear difference equation of the type

$$\Delta^2 y(x) + a(x) y(x+2) = 0, \tag{2}$$

in which $a(x)$ depends only upon a_0, a_1 and a_2 and is defined for all $x \geqq \alpha$.

In fact, if one places $u(x) = t(x)y(x)$, equation (1) becomes

$$a_0(x)t(x+2)y(x+2) + a_1(x)t(x+1)y(x+1) + a_2(x)t(x)y(x) = 0$$

or, since

$$y(x+1) = y(x+2) - \Delta y(x+1),$$

$$y(x) = y(x+2) - 2\Delta y(x+1) + \Delta^2 y(x),$$

the same equation becomes

$$\begin{aligned} &a_2(x)t(x)\Delta^2 y(x) - [a_1(x)t(x+1) + 2a_2(x)t(x)]\Delta y(x+1) \\ &\quad + [a_0(x)t(x+2) + a_1(x)t(x+1) + a_2(x)t(x)]y(x+2) = 0. \end{aligned} \tag{3}$$

Let us now choose the heretofore undetermined function $t(x)$ in such a manner that the second coefficient of (3) vanishes:

$$a_1(x)t(x+1) + 2a_2(x)t(x) = 0. \tag{4}$$

The function $t(x)$ thus becomes determined,† except for a constant factor, by the equation

$$t(x) = \prod_{x_1=\alpha}^{x_1=x-1} \left(\frac{-2a_2}{a_1}\right)_{x_1}, \tag{5}$$

since $a_1(x) \neq 0$ for $x \geqq \alpha$. Moreover, from the hypothesis $a_2(x) \neq 0$ it follows that $t(x) \neq 0$ and hence also $a_2(x)t(x) \neq 0$.

Consequently, equation (3) may be reduced to the form

$$\Delta^2 y(x) + [4(a_0/a_1)_x(a_2/a_1)_{x+1} - 1]\, y(x+2) = 0 \qquad (x \geqq \alpha), \tag{6}$$

i. e., to the desired form (2).

3. We shall now make the assumption that $4(a_0/a_1)_x(a_2/a_1)_{x+1} - 1$ for large values of x has the form $-\nu^2 - \phi(x) - \theta(x)$, where ν is a constant (real or

* Throughout the paper it will be understood, unless otherwise stated, that $x \geqq \alpha =$ a sufficiently large constant.

† See BOOLE's *Finite Differences*, Chap. IX, § 6.

complex) and $\nu(\nu^2-1)\neq 0$, where also $\phi(x)$ is a function of x of the form $\mu/x^{1+\delta}$, in which μ and δ are constants with $0<\delta\leqq\frac{1}{2}$, and where $\theta(x)$ is any function of x such that the series

$$\sum_{x_1=x}^{x_1=\infty}|\theta(x_1)|$$

converges. For example, the function $\theta(x)$ may be taken as any function of the form $g(x)\cdot\tau(x)/x$, where $|g(x)|<c=\text{const.}$ and $\tau(x)$ is one of the functions

$$\frac{1}{x^p},\quad \frac{1}{(\log x)^{1+p}},\quad \frac{1}{\log x(\log_2 x)^{1+p}},\cdots \qquad (p>0).$$

In particular, we observe that the above hypotheses are realized whenever the original equation (1) has coefficients $a_0(x)$, $a_1(x)$, $a_2(x)$ which are developable either in the form of convergent power series in $1/x$ $(x\geqq\alpha)$ or, more generally, as asymptotic series in $1/x$.

Equation (6) thus becomes

$$\Delta^2 y(x)-[\nu^2+\phi(x)+\theta(x)]y(x+2)=0. \tag{7}$$

In the study of this equation we shall now make use of Theorem II of the paper entitled *Sur les équations linéaires aux différences finies*,* using without further remark the results and notation there found.

In the present instance let us choose the auxiliary functions $z_1(x)$, $z_2(x)$ as follows, for reasons which will appear presently:

$$z_1(x)=\prod_{x_1=\alpha}^{x_1=x-1}\big(1+\nu+\phi(x_1)/2\nu\big),\qquad z_2(x)=\prod_{x_1=\alpha}^{x_1=x-1}\big(1-\nu-\phi(x_1)/2\nu\big). \tag{8}$$

Then

$$\Delta z_1(x)=\big(\nu+\phi(x)/2\nu\big)z_1(x),\qquad \Delta^2 z_1(x)=\big(\nu^2+\phi(x)+\xi_1(x)\big)z_1(x), \tag{9}$$

where

$$\xi_1(x)=\frac{1+\nu}{2\nu}\Delta\phi(x)+\frac{\phi(x+1)\phi(x)}{4\nu^2}.$$

From our hypothesis respecting $\phi(x)$ it appears directly that $\xi_1(x)$ has the character of one of the functions $\theta(x)$ mentioned above.

Likewise we have

$$\Delta z_2(x)=-\big(\nu+\phi(x)/2\nu\big)z_2(x),\qquad \Delta^2 z_2(x)=\big(\nu^2+\phi(x)+\xi_2(x)\big)z_2(x),$$

where $\xi_2(x)$ has the properties of $\xi_1(x)$ just mentioned and is obtained from it by changing ν into $-\nu$.

*Annali di Matematica, loc. cit., p. 301. The expression $\sum_{x_1=x+1}^{x_1=\infty}|u_m(x_1)|$ there occurring should be replaced by $\sum_{m=1}^{m=\infty}|u_m(x_1)|$.

Equations (8) and (9) together with those just noted give the following values for the expressions $A(x)$, $Q(x)$, $f_1(x)$, $f_2(x)$, $\bar{q}(x, x_1)$, $\Phi(x, x_1)$, $\Psi(x, x_1)$, and $Y(x)$ which occur in the statement of the above mentioned Theorem II:

$$Q(x) = -2\left(\nu + \phi(x)/2\nu\right) z_1(x) z_2(x), \tag{10}$$

$$A(x) = c_2 z_1(x) - c_1 z_2(x), \tag{11}$$

$$\begin{aligned} f_1(x) &= [-\nu^2 - \phi(x) - \theta(x)] z_1(x) + \Delta^2 z_1(x) = [\xi_1(x) - \theta(x)] z_1(x), \\ f_2(x) &= [\xi_2(x) - \theta(x)] z_2(x), \end{aligned} \tag{12}$$

$$\bar{q}(x, x_1) = \begin{vmatrix} z_1(x) & [\xi_1(x_1) - \theta(x_1)] z_1(x_1) \\ z_2(x) & [\xi_2(x_1) - \theta(x_1)] z_2(x_1) \end{vmatrix}, \tag{13}$$

$$\begin{aligned} \Phi(x, x_1) = {} & \frac{1}{2\left(\nu + \phi(x_1 + 1)/2\nu\right)} \\ & \times \left[\frac{[\xi_2(x_1) - \theta(x_1)] z_1(x+1)}{[1 - \nu - \phi(x_1)/2\nu] z_1(x_1+1)} - \frac{[\xi_1(x_1) - \theta(x_1)] z_2(x+1)}{[1 + \nu + \phi(x_1)/2\nu] z_2(x_1+1)} \right]. \end{aligned} \tag{14}$$

Moreover, since $\lim_{x_1 = \infty} \phi(x_1) = 0$, $\nu(1 - \nu^2) \neq 0$, while $\theta(x_1)$, $\xi_1(x_1)$, $\xi_2(x_1)$ have the properties above mentioned, we now see that

$$\Phi(x, x_1) = \frac{z_1(x+1)}{z_1(x_1+1)} \left[s_1(x_1) + s_2(x_1) \frac{z_1(x_1+1) z_2(x+1)}{z_1(x+1) z_2(x_1+1)} \right] |\theta(x_1)|, \tag{15}$$

where $|s_1(x_1)|$ and $|s_2(x_1)|$ are less than some assignable constant, and $\theta(x)$, though not identical with the $\theta(x)$ of (7), has the properties before described of that function.

Finally, we have

$$\Psi(x, x_1) = A(x_1 + 1) \Phi(x, x_1) \tag{16}$$

and

$$Y(x) = \frac{-1}{Q(x+1)} [A(x+1) + u_1(x) + u_2(x) + \cdots + u_m(x) + \cdots]. \tag{17}$$

III. *The special case* $|1 + \nu| = |1 - \nu|$.

4. Let us suppose in the first place that $|1 + \nu| = |1 - \nu|$ and that $|1 + \nu + \phi(x)/2\nu| = |1 - \nu - \phi(x)/2\nu| = \rho_x$, $(x \geqq \alpha)$. For this case $|z_1(x)| = |z_2(x)|$. Whence if we place

$$G(x) = \prod_{n=\alpha}^{n=x} \rho_n \tag{18}$$

we may write

$$\Phi(x, x_1) = \omega_1(x, x_1) |\theta(x_1)| \frac{G(x)}{G(x_1)} \qquad (x \geqq \alpha, x_1 > x), \tag{19}$$

where $|\omega_1(x, x_1)| < \Omega_1 = \text{const.}$

Similarly we have

$$A(x+1) = G(x)\,[c_1 r_1(x) + c_2 r_2(x)] \qquad (x \geqq \alpha), \tag{20}$$

where $|r_1(x)| = |r_2(x)| = 1$. Whence, by (16), it follows that

$$\Psi(x, x_1) = \omega_2(x, x_1)\,|\theta(x_1)|\,G(x) \qquad (x \geqq \alpha,\ x_1 > \alpha), \tag{21}$$

where $|\omega_2(x, x_1)| < \Omega_2 = \text{const.}$

Equations (19) and (21) having been obtained, we turn to consider the series

$$|u_1(x)| + |u_2(x)| + \cdots + |u_m(x)| + \cdots. \tag{22}$$

Referring to the definition of the term $u_m(x)$ given in the before mentioned Theorem II, we have in the present instance

$$\Phi(x, x_1)\,\Phi(x_1, x_2) \cdots \Phi(x_{m-2}, x_{m-1})\,\Psi(x_{m-1}, x_m) = G(x) P(x, x_1, x_2, \cdots, x_m), \tag{23}$$

where for $x_m \geqq x_{m-1} \geqq x_{m-2} \cdots \geqq x_1 \geqq x \geqq \alpha$ we may write

$$|P(x, x_1, x_2, \cdots, x_m)| < \Omega_1^{m-1}\,\Omega_2\,|\theta(x_1)\,\theta(x_2) \cdots \theta(x_m)|. \tag{24}$$

Relations (23) and (24) now enable us to show that the three conditions (a), (b), (c) of the theorem are here fulfilled. Condition (a) is fulfilled inasmuch as we have

$$\sum_{x_1=x+1}^{x_1=\infty} \sum_{x_2=x_1+1}^{x_2=\infty} \cdots \sum_{x_m=x_{m-1}+1}^{x_m=\infty} |\Phi(x, x_1)\,\Phi(x_1, x_2) \cdots \Phi(x_{m-2}, x_{m-1})\,\Psi(x_{m-1}, x_m)| < G(x)\,\Omega_1^{m-1}\,\Omega_2\,W(x),$$

where

$$W(x) = \sum_{x_1=x+1}^{x_1=\infty} |\theta(x_1)| \sum_{x_2=x_1+1}^{x_2=\infty} |\theta(x_2)| \cdots \sum_{x_m=x_{m-1}+1}^{x_m=\infty} |\theta(x_m)|,$$

and this expression has a meaning by virtue of our hypotheses concerning the function $\theta(x)$.

As to condition (b), let us put

$$\theta_1(x) = \sum_{x_1=x+1}^{x_1=\infty} |\theta(x_1)|. \tag{25}$$

Since $\theta_1(x)$ becomes arbitrarily small for all values of x sufficiently large, the term $u_m(x)$ takes the form

$$u_m(x) = G(x)\,V_m(x), \tag{26}$$

where

$$|V_m(x)| < \Omega_1^{m-1}\,\Omega_2\,|\theta_1(x)|^m. \tag{27}$$

Whence, if α be sufficiently large, we shall have

$$|u_m(x)| < G(x)\,k^m, \qquad k = \text{const.} < 1 \qquad (x \geqq \alpha). \tag{28}$$

Thus the series (22) converges for $x \geqq \alpha$ to a value $U(x)$ such that

$$U(x) = \gamma(x)\, G(x), \qquad \gamma(x) < \frac{k}{1-k}. \tag{29}$$

By virtue of (19) and (20) we may now write

$$|\Phi(x, x_1)|\, U(x_1) = \beta_1(x, x_1) G(x),$$

where

$$\beta_1(x, x_1) < \Omega_2 |\theta(x_1)| \qquad (\Omega_2 = \text{const.},\ x \geqq \alpha,\ x_1 \geqq \alpha).$$

Thus the series

$$\sum_{x_1=x+1}^{x_1=\infty} |\Phi(x, x_1)|\, U(x_1) \qquad (x \geqq \alpha)$$

converges.

Finally, the expressions $Y(x_1) f_r(x_1)$, $(r = 1, 2)$, of condition (c) are to be considered. We obtain in the first place from (17) and (29)

$$Y(x) = -\left(\frac{A(x+1)}{Q(x+1)} + \frac{\beta_2(x) G(x)}{Q(x+1)}\right) \qquad \left(|\beta_2(x)| < \gamma(x) < \frac{k}{1-k}\right). \tag{30}$$

But from (10) and (11) we have

$$\frac{A(x+1)}{Q(x+1)} = -\frac{\nu}{2\nu^2 + \phi(x+1)}\left(\frac{c_2}{z_2(x+1)} - \frac{c_1}{z_1(x+1)}\right) = \frac{\beta_3(x)}{G(x)} \qquad (x \geqq \alpha), \tag{31}$$

where $|\beta_3(x)| < \beta_3 = \text{const.}$

Similarly we obtain

$$\frac{G(x)}{Q(x+1)} = \frac{\beta_4(x)}{G(x)} \qquad (|\beta_4(x)| < \beta_4 = \text{const.}). \tag{32}$$

Whence follows

$$Y(x) = \frac{\beta_5(x)}{G(x)} \qquad (|\beta_5(x)| \leqq \beta_5 = \text{const.}).$$

Moreover, we have from (12),

$$f_r(x) = \beta_6(x, r) G(x) |\theta(x)| \qquad (r = 1, 2),$$

where $|\beta_6(x, r)| < \beta_6 = \text{const.}$

Thence, noting that $\rho_\alpha \neq 0$, we have

$$Y(x) f_r(x) = \beta_7(x, r) |\theta(x)| \quad (|\beta_7(x, r)| < \beta_7 = \text{const.}),$$

and, therefore, the expressions

$$\sum_{x_1=x+1}^{x_1=\infty} Y(x_1) f_r(x_1) \qquad (r = 1, 2;\ x \geqq \alpha),$$

have a meaning. Thus all the conditions of Theorem II become fulfilled.

Moreover, the function $\gamma(x)$ of (29) has the properties of the function $\theta_1(x)$

defined in (25); i. e., $\lim_{x=\infty}\gamma(x)=0$. Consequently from (30), (31) and (32) we have

$$Y(x)=\frac{\nu}{2\nu^2+\phi(x+1)}\left(\frac{c_2}{z_2(x+1)}-\frac{c_1}{z_1(x+1)}+\frac{\delta(x)}{G(x)}\right) \tag{33}$$

where $x\geqq\alpha$, $\lim_{x=\infty}\delta(x)=0$.

Upon applying now the result of Theorem II we find that the general solution of (7), under the present hypotheses respecting ν, $\phi(x)$, and $\theta(x)$, will have the following form when $x\geqq\alpha$ for sufficiently large α:

$$y(x)=\frac{k_1\left(1+\epsilon_1(x)\right)}{z_1(x-1)}+\frac{k_2\left(1+\epsilon_2(x)\right)}{z_2(x-1)} \qquad \left(\lim_{x=\infty}\epsilon_1(x)=\lim_{x=\infty}\epsilon_2(x)=0\right),$$

k_1, k_2 being arbitrary constants. Since also

$$\frac{z_1(x)}{z_1(x-1)}=1+\eta_1(x), \qquad \frac{z_2(x)}{z_2(x-1)}=1+\eta_2(x) \qquad \left(\lim_{x=\infty}\eta_1(x)=\lim_{x=\infty}\eta_2(x)=0\right),$$

this may be thrown into the form

$$y(x)=\frac{k_1\left(1+\epsilon_1(x)\right)}{z_1(x)}+\frac{k_2\left(1+\epsilon_2(x)\right)}{z_2(x)} \qquad \left(\lim_{x=\infty}\epsilon_1(x)=\lim_{x=\infty}\epsilon_2(x)=0\right). \tag{34}$$

IV. *Derivation of a first integral for the general case.*

5. We turn now to consider the cases in which $|1+\nu|\neq|1-\nu|$. Placing $\rho_x=|1+\nu+\phi(x)/2\nu|$ as before and also placing $\sigma_x=|1-\nu-\phi(x)/2\nu|$ and

$$H(x)=\prod_{n=a}^{n=x}\sigma_n, \tag{35}$$

we shall have

$$\frac{z_2(x+1)}{z_1(x+1)}=\lambda_3(x,x_1)\frac{H(x)}{H(x_1)} \qquad (|\lambda_3(x,x_1)|=1).$$

Thus, from (15) we obtain

$$\Phi(x,x_1)=\frac{H(x)}{H(x_1)}\left(s_1(x_1)\frac{G(x)H(x_1)}{G(x_1)H(x)}+s_2(x_1)\right)|\theta(x_1)|.$$

Then

$$\frac{G(x)H(x_1)}{G(x_1)H(x)}=\left(\frac{\sigma_{x_1}}{\rho_{x_1}}\right)\left(\frac{\sigma_{x_1-1}}{\rho_{x_1-1}}\right)\cdots\left(\frac{\sigma_{x+1}}{\rho_{x+1}}\right).$$

Let us now suppose $x_1>x\geqq\alpha$ and let us consider ν to be that square root of ν^2 in (7) for which $|1+\nu|>|1-\nu|$. Then the factors in the right hand member of our last equation will each be less than 1 and we shall be able to write

$$\Phi(x,x_1)=\omega_3(x,x_1)|\theta(x_1)|\frac{H(x)}{H(x_1)} \qquad (|\omega_3(x,x_1)|<\Omega_3=\text{const.}). \tag{36}$$

As regards the function

$$\Psi(x, x_1) = A(x_1+1)\Phi(x, x_1) = [c_2 z_1(x_1+1) - c_1 z_2(x_1+1)]\,\Phi(x, x_1),$$

let us take in the present instance $c_2 = 0$. Then

$$A(x+1) = c_1 r_1(x) H(x) \qquad (|r(x)|=1),$$

so that

$$\Psi(x, x_1) = \omega_4(x, x_1) H(x) |\theta(x_1)| \qquad (|\omega_4(x, x_1)| < \Omega_4 = \text{const.}). \tag{37}$$

We may now proceed as in the former case. Thus we have in the first place

$$u_m(x) = H(x) V_m(x) \qquad (x \geqq a), \tag{38}$$

where $V_m(x)$ has the properties indicated in (27), whence also

$$|u_m(x)| < H(x) k^m \qquad (k = \text{const.} < 1;\ x \geqq a), \tag{39}$$

and

$$Y(x) = \frac{\nu}{2\nu^2 + \phi(x+1)} \left(\frac{-c_1}{z_1(x+1)} + \frac{\delta_1(x)}{G(x)} \right) \quad (\lim_{x=\infty} \delta_1(x) = 0).$$

Thus in place of (34) we now have the solution

$$y_1(x) = \frac{k_1(1+\epsilon_1(x))}{z_1(x)} \qquad (x \geqq a,\ \lim_{x=\infty} \epsilon_1(x) = 0). \tag{40}$$

V. *Derivation of the second integral.*

6. Having obtained but a particular solution $y_1(x)$ of (7) when $|1+\nu| \neq |1-\nu|$, we proceed in the present section to obtain a second solution and therewith the general solution.

For this purpose equation (7) may be written in the form

$$\big(1 - \nu^2 - \phi(x) - \theta(x)\big) y(x+2) - 2y(x+1) + y(x) = 0. \tag{41}$$

If now $u(x)$ and $v(x)$ be any two linearly independent solutions of the equation

$$a_0(x) u(x+2) + a_1(x) u(x+1) + a_2(x) u(x) = 0, \tag{42}$$

we have, after placing for brevity $a_0(x) = a_0$, etc., and $u(x+1) = u_1$, $u(x+2) = u_2$, etc.:

$$a_0(u_2 v_1 - u_1 v_2) + a_2(u_0 v_1 - u_1 v_0) = 0$$

and hence

$$\frac{u_2 v_1 - u_1 v_2}{u_1 v_0 - u_0 v_1} = \frac{a_2}{a_0}.$$

Therefore, for any fixed integer α we obtain

$$u(\alpha+n)v(\alpha+n-1)-u(\alpha+n-1)v(\alpha+n)=c\prod_{p=a}^{p=a+n-2}\frac{a_2(p)}{a_0(p)} \qquad (n=2,3,\cdots),$$

where c is a constant as regards n. Whence

$$\frac{u(\alpha+n)}{v(\alpha+n)}-\frac{u(\alpha+1)}{v(\alpha+1)}=c\sum_{m=2}^{m=n}\frac{1}{v(\alpha+m)v(\alpha+m-1)}\prod_{p=a}^{p=a+m-2}\frac{a_2(p)}{a_0(p)}.$$

Thus, if $v(x)$ be looked upon as a *known* solution of (42), the solution $u(x)$ may be expressed in the following form for all values of $x=\alpha+n$, α being a constant:

(43) $$u(x)=c_1v(x)+c_2v(x)S_n$$

where

$$S_n=\sum_{m=1}^{m=n}\frac{1}{v(\alpha+m)v(\alpha+m-1)}\prod_{p=a}^{p=a+m-2}\frac{a_2(p)}{a_0(p)}.$$

We proceed to study the properties of S_n for large values of n in the case of the special equation (41), for which one solution $v(x)\equiv y_1(x)$, given by (40), is known.

Thus we have in the present instance

(44) $$\frac{1}{v(\alpha+m)}=k_1(1+\epsilon_m)z_1(\alpha+m)=k_1(1+\epsilon_m)\prod_{p=a}^{p=a+m-1}\left(1+\nu+\frac{\phi(p)}{2\nu}\right) \qquad (\lim_{m=\infty}\epsilon_m=0),$$

and from (41)

$$\frac{a_2(p)}{a_0(p)}=\frac{1+\eta_p}{\left(1+\nu+\frac{\phi(p)}{2\nu}\right)\left(1-\nu-\frac{\phi(p)}{2\nu}\right)},$$

where

$$\eta_p=\frac{\phi(p)-\left[\frac{\phi(p)}{2\nu}\right]^2}{1-\nu^2-\phi(p)-\theta(p)}.$$

Thence,

(45) $$\prod_{p=a}^{p=a+m-2}\frac{a_2(p)}{a_0(p)}=\frac{\prod_{p=a}^{p=a+m-2}(1+\eta_p)}{\prod_{p=a}^{p=a+m-2}\left(1+\nu+\frac{\phi(p)}{2\nu}\right)\left(1-\nu-\frac{\phi(p)}{2\nu}\right)}.$$

Moreover, by virtue of our hypotheses upon $\phi(x)$ and $\theta(x)$, the series

$$\sum_{p=a}^{p=\infty}|\eta_p|$$

is convergent, and hence it follows that

(46) $$\prod_{p=a}^{p=a+m-2}(1+\eta_p)=k_2(1+\epsilon_m) \qquad (k_2=\text{const.},\ \lim_{m=\infty}\epsilon_m=0).$$

Upon making use of (44), (45) and (46) we find that the mth term of S_n is of the form

$$k_2(1+\epsilon_m)\frac{\prod_{p=a}^{p=a+m-1}\left(1+\nu+\frac{\phi(p)}{2\nu}\right)}{\prod_{p=a}^{p=a+m-2}\left(1-\nu-\frac{\phi(p)}{2\nu}\right)}=\frac{k_2'(1+\epsilon_m')z_1(\alpha+m)}{z_2(\alpha+m)} \qquad (\lim_{m=\infty}\epsilon_m'=0).$$

The expression S_n may therefore be put into the form

$$S_n = k_2'\frac{z_1(\alpha+n)}{z_2(\alpha+n)}S_n', \tag{47}$$

where

$$S_n' = \sum_{m=1}^{m=n}\frac{z_1(\alpha+m)z_2(\alpha+n)}{z_1(\alpha+n)z_2(\alpha+m)}(1+\epsilon_m'),$$

which we may now show directly to be of the form $k_3+\epsilon_n$ where $k_3=$ const. and $\lim_{n=\infty}\epsilon_n=0$. In fact, we may write

$$S_n' = \left(\sum_{m=1}^{m=n-p-1}+\sum_{m=n-p}^{m=n}\right)\frac{z_1(\alpha+m)z_2(\alpha+n)}{z_1(\alpha+n)z_2(\alpha+m)}(1+\epsilon_m),$$

where the first sum appearing on the right evidently approaches a limit by virtue of (8) when $n=\infty$, while by taking p sufficiently large the second sum may be made less in absolute value than any pre-assigned positive quantity ϵ, whatever n may be $(n>p+1)$, since

$$\left|\frac{z_1(\alpha+m)z_2(\alpha+n)}{z_1(\alpha+n)z_2(\alpha+m)}\right| = \frac{\sigma_{a+n}\sigma_{a+n-1}\cdots\sigma_{a+m-1}}{\rho_{a+n}\rho_{a+n-1}\cdots\rho_{a+m-1}} < q^{n-m+2} \qquad (q=\text{const.}<1).$$

Availing ourselves of the form just established for S_n' and recalling the definition of $v(\alpha+n)$, we see from (47) that

$$S_n = \frac{k(1+\epsilon_n)}{v(\alpha+n)z_2(\alpha+n)} = \frac{k\big(1+\epsilon(x)\big)}{v(x)z_2(x)} \qquad (\lim_{n=\infty}\epsilon_n=\lim_{x=\infty}\epsilon(x)=0).$$

Placing this value in (43) and recalling that the expression $c_1v(x)=c_1y_1(x)$ there appearing is itself a solution of (41), we obtain the following second solution linearly independent of $y_1(x)$:

$$y_2(x)=\frac{k_2\big(1+\epsilon_2(x)\big)}{z_2(x)} \qquad (x\geqq\alpha,\ \lim_{x=\infty}\epsilon_2(x)=0).$$

Our results may now be summarized in the following general theorem:

Theorem I: *Given the equation*

$$a_0(x)u(x+2)+a_1(x)u(x+1)+a_2(x)u(x)=0$$

whose coefficients (real or complex) are defined for all positive integral values of x sufficiently large ($x\geqq\alpha$), and satisfy for such values the conditions:

(1) $a_0(x)$, $a_1(x)$, $a_2(x)$ *never vanish.*

(2) *The expression*

$$4\frac{a_0(x)a_2(x+1)}{a_1(x)a_1(x+1)}-1$$

has the form $-\nu^2-\mu/x^{1+\delta}-\theta(x)$ *where* μ, ν, δ *are constants such that* $0<\delta\leqq\frac{1}{2}$ *and* $\nu(1-\nu^2)\neq 0$ *and where* $\theta(x)$ *is any function of* x *such that the series*

$$\sum_{x_1=x}^{x_1=\infty}|\theta(x_1)| \qquad (x\geqq\alpha)$$

converges.

Then if $|1+\nu|\neq|1-\nu|$, *there are two particular solutions of the equation having the forms*

$$u_1(x)=\frac{t(x)\big(1+\epsilon_1(x)\big)}{z_1(x)},\quad u_2(x)=\frac{t(x)\big(1+\epsilon_2(x)\big)}{z_2(x)} \qquad (\lim_{x=\infty}\epsilon_1(x)=\lim_{x=\infty}\epsilon_2(x)=0),$$

in which

$$t(x)=\prod_{x_1=\alpha}^{x_1=x-1}\left(\frac{-2a_2}{a_1}\right)_{x_1},\quad z_1(x)=\prod_{x_1=\alpha}^{x_1=x-1}\left\{1+\nu+\frac{\phi(x_1)}{2\nu}\right\},\quad z_2(x)=\prod_{x_1=\alpha}^{x_1=x-1}\left\{1-\nu-\frac{\phi(x_1)}{2\nu}\right\}.$$

When $|1+\nu|=|1-\nu|$ *the result continues to hold true provided that*

$$\left|1+\nu+\frac{\phi(x)}{2\nu}\right|=\left|1-\nu-\frac{\phi(x)}{2\nu}\right| \text{ for } x\geqq\alpha.$$

VI. *Application to the equation considered by Horn.**

7. We proceed to apply the above theorem to the study of the solutions $u(x)$ of the equation,

$$P_0(x)u(x+2)+x^kP_1(x)u(x+1)+x^{2k}P_2(x)u(x)=0, \tag{48}$$

where k is any integer, positive, negative or zero and $P_0(x)$, $P_1(x)$ and $P_2(x)$ are either convergent series for all sufficiently large values of x or are developable asymptotically in the form

$$P_\lambda(x)=a_\lambda+\frac{b_\lambda}{x}+\frac{c_\lambda}{x^2}+\cdots+\frac{p_\lambda+\omega_\lambda(x)}{x^n} \qquad (\lim_{x=\infty}\omega_\lambda(x)=0).$$

For this purpose we begin by making the transformation $u(x)=[\Gamma(x)]^k y(x)$. Thus equation (48) takes the form

$$\begin{aligned}&\left(a_0+\frac{b_0}{x}+\frac{c_0}{x^2}+\cdots\right)y(x+2)\\ &+\left(a_1'+\frac{b_1'}{x}+\frac{c_1'}{x^2}+\cdots\right)y(x+1)+\left(a_2'+\frac{b_2'}{x}+\cdots\right)y(x)=0,\end{aligned} \tag{49}$$

* Loc. cit., p. 190.

in which the coefficients of $y(x+1)$ and $y(x)$ are of the form just indicated for $P_\lambda(x)$. In particular,

$$(50) \qquad a_1' = a_1, \qquad b_1' = b_1 - a_1 k, \qquad a_2' = a_2, \qquad b_2' = b_2 - a_2 k.$$

For convenience, let us drop the primes in (49) and take as object of study the equation

$$(51) \quad \left(a_0 + \frac{b_0}{x} + \cdots\right) y(x+2) + \left(a_1 + \frac{b_1}{x} + \cdots\right) y(x+1) + \left(a_2 + \frac{b_2}{x} + \cdots\right) y(x) = 0,$$

in which the coefficients are of the type $P_\lambda(x)$.

Equation (4) becomes in the present instance

$$(52) \quad \left(a_1 + \frac{b_1}{x} + \frac{c_1}{x^2} + \cdots\right) t(x+1) + 2\left(a_2 + \frac{b_2}{x} + \frac{c_2}{x^2} + \cdots\right) t(x) = 0.$$

Place $t(x) = x^g(1 + h/x)v(x)$ where g and h are constants yet to be determined. Equation (52) then takes the form

$$\left(1 + \frac{h}{x+1}\right)\left(a_1 + \frac{b_1}{x} + \cdots\right) v(x+1) + 2\left(1 + \frac{1}{x}\right)^{-g}\left(1 + \frac{h}{x}\right)\left(a_2 + \frac{b_2}{x} + \cdots\right) v(x) = 0,$$

or, upon developing $(1 + 1/x)^{-g}$ by the binomial theorem,

$$(53) \quad \left(a_1 + (b_1 + ha_1)\frac{1}{x} + (c_1 + hb_1 - ha_1)\frac{1}{x^2} + \cdots\right) v(x+1) + 2\left(a_2 + (b_2 - ga_2 + ha_2)\frac{1}{x} + (c_2 - gb_2 + hb_2 + \tfrac{1}{2}g(g+1)a_2 - gha_2)\frac{1}{x^2} + \cdots\right) v(x) = 0.$$

Let us now choose the undetermined constants g, h so that the term in $1/x$ in each coefficient of (53) vanishes. In case $a_1 \neq 0$, $a_2 \neq 0$, these two conditions determine g and h as follows:

$$g = \frac{a_1 b_2 - a_2 b_1}{a_1 a_2}, \qquad h = -\frac{b_1}{a_1}.$$

The constants g, h having been thus determined, we may now apply directly to equation (53) the results embodied in Theorem III of my previous paper* and write for sufficiently large α

$$v(x) = c_1\left(-\frac{2a_2}{a_1}\right)^x (1 + \epsilon_1(x)) \qquad (c_1 = \text{const.}\ x \geqq \alpha,\ \lim_{x=\infty} \epsilon_1(x) = 0),$$

and therefore also

$$(54) \qquad t(x) = c_1 x^{\frac{a_1 b_2 - a_2 b_1}{a_1 a_2}}\left(-\frac{2a_2}{a_1}\right)^x (1 + \epsilon_2(x)) \qquad (\lim_{x=\infty} \epsilon_2(x) = 0).$$

* Loc. cit., p. 313.

We turn now to consider the forms taken by $z_1(x)$, $z_2(x)$ for the equation (51). In the first place, let us construct the expression

$$\frac{4a_0(x)a_2(x+1)}{a_1(x)a_1(x+1)}-1$$

referred to in condition (b) of the preceding theorem. Since

$$\frac{a_0(x)}{a_1(x)}=\frac{a_0}{a_1}+\frac{a_1b_0-a_0b_1}{a_1^2}\left(\frac{1}{x}\right)+\cdots,$$

$$\frac{a_2(x+1)}{a_1(x+1)}=\frac{a_2}{a_1}+\frac{a_1b_2-a_2b_1}{a_1^2}\left(\frac{1}{x+1}\right)+\cdots,$$

we obtain

$$4\,\frac{a_0(x)a_2(x+1)}{a_1(x)a_1(x+1)}-1=-\nu^2-\frac{\mu}{x+1}-\theta(x),$$

in which

$$\nu=\frac{\sqrt{a_1^2-4a_0a_2}}{a_1}, \tag{55}$$

$$\mu=4\,\frac{2a_0a_2b_1-a_1a_2b_0-a_0a_1b_2}{a_1^3}, \tag{56}$$

and $\theta(x)$ vanishes to at least the second order when $x=\infty$.

Thus, for the function $z_1(x)$ we have

$$z_1(x)=\prod_{x_1=a}^{x_1=x}\left(1+\nu+\frac{\mu}{2\nu x_1}\right)=g_1(1+\nu)^{x+1}\,\frac{\Gamma\left(x+1+\dfrac{\mu}{2\nu(1+\nu)}\right)}{\Gamma(x+1)} \tag{57}$$

where g_1 is a constant and μ and ν are given by (56) and (55). Similarly,

$$z_2(x)=g_2(1-\nu)^{x+1}\,\frac{\Gamma\left(x+1-\dfrac{\mu}{2\nu(1-\nu)}\right)}{\Gamma(x+1)}. \tag{58}$$

Let us next consider what conditions (a) and (b) of Theorem I become in the present instance. Since

$$\nu(\nu^2-1)=\frac{4a_0a_2}{a_1^3}\sqrt{a_1^2-4a_0a_2},$$

they will be satisfied if $a_0a_1a_2(a_1^2-4a_0a_2)\neq 0$. Moreover, the roots of the quadratic equation $a_0\lambda^2+a_1\lambda+a_2=0$ are

$$\lambda_1=(1/2a_0)(-a_1+\sqrt{a_1^2-4a_0a_2}),\qquad \lambda_2=(1/2a_0)(-a_1-\sqrt{a_1^2-4a_0a_2}),$$

so that

$$1+\nu=-\frac{2a_2}{a_1\lambda_1},\qquad 1-\nu=-\frac{2a_2}{a_1\lambda_2}.$$

Also we have

$$\frac{\mu}{2\nu(1+\nu)} = \frac{\mu'}{a_0 a_1(a_1\lambda_2 + 2a_2)}, \qquad \frac{-\mu}{2\nu(1-\nu)} = \frac{\mu'}{a_0 a_1(a_1\lambda_1 + 2a_2)},$$

where $\mu' = -\mu a_1^3/4 = a_0 a_1 b_2 + a_1 a_2 b_0 - 2a_0 a_2 b_1$.

Noting that $a_1^2 - 4a_0 a_2 \neq 0$ whenever $\lambda_1 \neq \lambda_2$, we see that if in equation (51) the roots λ_1, λ_2 of the quadratic $a_0\lambda^2 + a_1\lambda + a_2 = 0$ are of unequal modulus and $a_0 a_1 a_2 \neq 0$, there are two solutions of the same equation which, when considered for all positive integral values of x sufficiently large, take the forms

$$(59) \quad y_1(x) = x^g \frac{\Gamma(x)\lambda_1^x\big(1+\epsilon_1(x)\big)}{\Gamma\left(x + \dfrac{p}{q\lambda_2 + r}\right)}, \qquad y_2(x) = x^g \frac{\Gamma(x)\lambda_2^x\big(1+\epsilon_2(x)\big)}{\Gamma\left(x + \dfrac{p}{q\lambda_1 + r}\right)},$$

wherein $\lim_{x=\infty} \epsilon_1(x) = \lim_{x=\infty} \epsilon_2(x) = 0$, and the constants have the values

$$g = \frac{a_1 b_2 - a_2 b_1}{a_1 a_2}, \quad p = a_0 a_1 b_2 + a_1 a_2 b_0 - 2a_0 a_2 b_1, \quad q = a_0 a_1^2, \quad r = 2a_0 a_1 a_2.$$

Moreover, for the cases in which λ_1, λ_2 are distinct but of equal modulus we see from (57) and (58) that the result just obtained will continue to hold true, by Theorem I, provided that for all $x \geqq \alpha$ we have

$$|x + \mu/2\nu(1+\nu)| = |x - \mu/2\nu(1-\nu)|,$$

i. e. $|x + c_1| = |x + c_2|$ where

$$c_1 = \frac{p}{q\lambda_2 + r}, \qquad c_2 = \frac{p}{q\lambda_1 + r}.$$

But if this latter condition is satisfied, it is evident that c_1 and c_2 are conjugate imaginaries, and conversely. Thus the result already obtained when $|\lambda_1| \neq |\lambda_2|$ will continue to hold true when $|\lambda_1| = |\lambda_2|$, provided that $\lambda_1 \neq \lambda_2$, and either c_1, c_2 are conjugate imaginaries or $c_1 = c_2 = 0$.

We note also that the form of the solutions (59) may be somewhat simplified by making use of the well known asymptotic relation

$$\Gamma(x) \sim \sqrt{2\pi} e^{-x} x^{x-\frac{1}{2}}.$$

Thus for any constant l we have

$$\frac{\Gamma(x+l)}{\Gamma(x)} \sim e^{-l}(x+l)^l \left(1 + \frac{l}{x}\right)^{x-\frac{1}{2}} \sim e^{-l} x^l \left(1 + \frac{l}{x}\right)^x \sim x^l,$$

so that relations (59) may be replaced by

$$(60) \qquad y_1(x) = x^{h_1}\lambda_1^x\big(1+\epsilon_1(x)\big), \qquad y_2(x) = x^{h_2}\lambda_2^x\big(1+\epsilon_2(x)\big),$$

wherein

$$h_1 = g - c_1 = \frac{a_1 b_2 - a_2 b_1}{a_1 a_2} - \frac{p}{q\lambda_2 + r}, \qquad h_2 = \frac{a_1 b_2 - a_2 b_1}{a_1 a_2} - \frac{p}{q\lambda_1 + r}.$$

8. Besides the cases already considered in which $a_0 a_1 a_2 \neq 0$ it is deserving of note that whenever $a_0 a_2 \neq 0$, $a_1 = b_1 = 0$, the nature of the solutions of (51) for large positive values of x may still be found by the application of known results. For after making the transformation

$$y(x) = x^g \left(1 + \frac{h}{x}\right) v(x)$$

the same equation takes the form

$$A_0(x)v(x+2) + A_1(x)v(x+1) + A_2(x)v(x) = 0,$$

where

$$A_0(x) = a_0 + (b_0 + ha_0)\frac{1}{x} + \cdots,$$

$$A_1(x) = \frac{c_1}{x^2} + \cdots,$$

$$A_2(x) = a_2 + (b_2 - 2ga_2 + ha_2)\frac{1}{x} + \cdots.$$

Then by choosing h and g so that the coefficients of $1/x$ here appearing vanish, i. e.,

$$g = \frac{a_0 b_2 - a_2 b_0}{2a_0 a_2}, \qquad h = -\frac{b_0}{a_0},$$

we may at once apply to equation (51) the Theorem III of the aforesaid memoir. As the roots of the equation $a_0\lambda^2 + a_2 = 0$ are unequal but of equal modulus, we conclude that there are two solutions under the present hypotheses having respectively the forms

$$(61) \quad y_1(x) = x^g \lambda_1^x \big(1 + \epsilon_1(x)\big), \qquad y_2(x) = x^g \lambda_2^x \big(1 + \epsilon_2(x)\big) \quad \big(\lim_{x=\infty} \epsilon_1(x) = \lim_{x=\infty} \epsilon_2(x) = 0\big),$$

where

$$g = (a_0 b_2 - a_2 b_0)/2a_0 a_2.$$

9. Returning now to the original equation (48) and recalling that

$$u(x)[\Gamma(x)]^k y(x) = \left(\frac{x}{e}\right)^{kx} x^{-\frac{1}{2}k}(2\pi)^{\frac{1}{2}k} y(x)\big(1 + \epsilon(x)\big),$$

where $\lim_{x=\infty} \epsilon(x) = 0$, and making use of equations (50) we reach in summary the following theorem:

THEOREM II. *Given the equation*

$$P_0(x)u(x+2) + x^k P_1(x)u(x+1) + x^{2k} P_2(x)u(x) = 0,$$

where k is any integer, positive, negative or zero, while $P_0(x)$, $P_1(x)$, $P_2(x)$

are convergent series for sufficiently large values of x, or are any functions of x developable asymptotically in the form

$$P_\lambda(x) = a_\lambda + \frac{b_\lambda}{x} + \frac{c_\lambda}{x^2} + \cdots + \frac{p_\lambda + \omega_\lambda(x)}{x^n} \qquad (\lim_{x=\infty} \omega_\lambda(x) = 0).$$

CASE I. *If $a_0 a_1 a_2 \neq 0$ and if the roots λ_1, λ_2 of the quadratic equation $a_0\lambda^2 + a_1\lambda + a_2 = 0$ are of unequal modulus, there are two solutions of the given equation which, when considered for all positive integral values of x sufficiently large, take the respective forms*

$$u_1(x) = \left(\frac{x}{e}\right)^{kx} x^{\rho+\sigma_2}\lambda_1^x\left(1 + \epsilon_1(x)\right), \quad u_2(x) = \left(\frac{x}{e}\right)^{kx} x^{\rho+\sigma_1}\lambda_2^x\left(1 + \epsilon_2(x)\right)$$
$$(\lim_{x=\infty} \epsilon_1(x) = 0, \lim_{x=\infty} \epsilon_2(x = 0)).$$

wherein ρ, σ_1, σ_2 are constants defined as follows:

$$\rho = \frac{a_1 b_2 - a_2 b_1}{a_1 a_2} - \frac{k}{2},$$

$$\sigma_i = \frac{2a_0 a_2 b_i - a_0 a_1 b_2 - a_1 a_2 b_0 - k a_0 a_1 a_2}{a_0 a_1 (a_1\lambda_i + 2a_2)} \qquad (i = 1, 2).$$

Moreover, the same result holds true whenever the roots λ_1, λ_2 are unequal but of equal modulus, provided that σ_1 and σ_2 are conjugate imaginaries, including the case in which $\sigma_1 = \sigma_2 = 0$.

CASE II. *If $a_0 a_2 \neq 0$, $a_1 = b_1 = 0$ and if we represent by λ_1, λ_2 the roots of the quadratic equation $a_0\lambda^2 + a_2 = 0$, there are two solutions of the given equation which when considered for values of x sufficiently large take the forms*

$$u_1(x) = \left(\frac{x}{e}\right)^{kx} x^{\rho}\lambda_1^x\left(1 + \epsilon_1(x)\right), \qquad u_2(x) = \left(\frac{x}{e}\right)^{kx} x^{\rho}\lambda_2^x\left(1 + \epsilon_2(x)\right)$$
$$(\lim_{x=\infty} \epsilon_1(x) = \lim_{x=\infty} \epsilon_2(x) = 0),$$

where ρ is defined by the relation

$$\rho = \frac{a_0 b_2 - a_2 b_0}{2a_0 a_2} - \frac{k}{2}.$$

10. The results obtained under Case I of the above theorem are in accord with those obtained by HORN.* To show this we evidently need to show merely that our values of $\rho + \sigma_1$, $\rho + \sigma_2$ are equal respectively to the quantities ρ_1, ρ_2 employed by HORN, and defined† by the relation

$$\rho_1 = -\frac{k(a_1 + 4a_0\lambda_1)}{2(a_1 + 2a_0\lambda_1)} - \frac{b_2 + b_1\lambda_1 + b_0\lambda_1^2}{a_1\lambda_1 + 2a_0\lambda_1^2},$$

with a similar formula for ρ_2 obtained by replacing λ_1 by λ_2.

* Loc. cit., p. 192.

† Loc. cit., p. 191, footnote.

Now in the sum $\rho + \sigma_1$ the coefficient of $\frac{1}{2}k$ is $-1 - 2a_2/(a_1\lambda_2 + 2a_2)$. But the coefficient of $-\frac{1}{2}k$ in ρ_1 may be written in the form

$$-1 - \frac{2a_0\lambda_1}{a_1 + 2a_0\lambda_1} = -1 - \frac{2a_0\lambda_1\lambda_2}{a_1\lambda_1 + 2a_0\lambda_1\lambda_2},$$

which becomes the same as the coefficient of $-\frac{1}{2}k$ in $\rho + \sigma_1$ since $\lambda_1\lambda_2 = a_2/a_0$. It remains only to show that

$$\frac{a_1b_2 - a_2b_1}{a_1a_2} + \frac{2a_0a_2b_1 - a_0a_1b_2 - a_1a_2b_0}{a_0a_1(a_1\lambda_2 + 2a_2)} = -\frac{b_2 + b_1\lambda_1 + b_0\lambda_1^2}{a_1\lambda_1 + 2a_0\lambda_1^2}.$$

This reduces to an identity by virtue of the relations

$$a_1\lambda_2 + 2a_2 = \frac{a_2}{a_0\lambda_1}(a_1 + 2a_0\lambda), \qquad a_0\lambda_1^2 + a_1\lambda_1 + a_2 = 0.$$

Similarly, by using λ_2 instead of λ_1 we obtain $\rho + \sigma_2 = \rho_2$.

It is to be observed that HORN's work concerns only case I.

11. As an illustration of the application of the preceding theorem, let us consider the equation

$$(62)\quad \left(1 + \frac{2}{x}\right)u(x+2) - \left(2 + \frac{3}{x}\right)z\,u(x+1) + \left(1 + \frac{1}{x}\right)u(x) = 0,$$

one of whose solutions is Legendre's function of the first kind $P_x(z)$. For simplicity, we shall confine ourselves to real values of z.

Here we have $k = 0$, $a_0 = 1$, $a_1 = -2z$, $a_2 = 1$, $b_0 = 2$, $b_1 = -3z$, $b_2 = 1$. Whence the roots λ_1, λ_2 are those of the quadratic $\lambda^2 - 2z\lambda + 1 = 0$; i. e., $\lambda_1 = z + \sqrt{z^2 - 1}$, $\lambda_2 = z - \sqrt{z^2 - 1}$. Thus, we shall have $|\lambda_1| \neq |\lambda_2|$ if $|z| > 1$, while we shall have $|\lambda_1| = |\lambda_2|$ but $\lambda_1 \neq \lambda_2$ if $|z| < 1$. Applying Case I of Theorem II, observing that $a_0a_1a_2 \neq 0$ when $z \neq 0$ and that in the present instance $\rho = -\frac{1}{2}$, $\sigma_1 = \sigma_2 = 0$, we find that for all real values of z except $z = 0$ the general solution of the above equation, when considered for all positive integral values of x sufficiently large takes the form

$$u(x) = \frac{1}{\sqrt{x}}\left[k_1(z + \sqrt{z^2 - 1})^x\big(1 + \epsilon_1(x)\big) + k_2(z - \sqrt{z^2 - 1})^x\big(1 + \epsilon_2(x)\big)\right]$$

$$\left(\lim_{x=\infty}\epsilon_1(x) = \lim_{x=\infty}\epsilon_2(x) = 0\right).$$

k_1, k_2 being arbitrary constants.

Moreover, precisely the same result holds when $z = 0$, as appears directly by applying Case II of the same theorem.

If, in particular, $-1 < z < 1$, we may place $z = \cos\xi$ and write $(z + \sqrt{z^2 - 1})^x = \cos x\xi + i\sin x\xi$. The solution $u(x)$ then takes the form

$$u(x) = \frac{1}{\sqrt{x}}\Big[k_1\big(1 + \epsilon_1(x)\big)\cos x\xi + k_2\big(1 + \epsilon_2(x)\big)\sin x\xi\Big],$$

where k_1, k_2, ϵ_1, ϵ_2 have the properties already mentioned. Moreover, upon determining two constants λ, μ from the equations $\lambda \sin \mu = k_1$, $\lambda \cos \mu = k_2$, and inserting for the constants k_1, k_2 these expressions, we obtain for $u(x)$ the form

$$u(x) = \frac{\lambda}{\sqrt{x}} [\sin (x\xi + \mu) + \epsilon(x)] \qquad (\lim_{x=\infty} \epsilon(x) = 0)$$

where λ and μ are arbitrary constants.

This last result for the special case in which $u(x) = P_x(z)$ agrees with other well known results respecting the behavior of Legendre's function of the first kind for large values of x. Previous investigations upon the subject, however, appear to have been from the standpoint of the *differential* equation satisfied by $P_x(z)$ rather than from that of the *difference* equation (62).*

*Cf. DINI, *Studi sulle equazioni differenziali*, Annali di Matematica, ser. 3, vol. 3 (1899), p. 178.

ON CANTOR'S THEOREM CONCERNING THE COEFFICIENTS OF A CONVERGENT TRIGONOMETRIC SERIES, WITH GENERALIZATIONS*

BY

WILLIAM F. OSGOOD

§ 1. CANTOR'S THEOREM.

CANTOR† has shown that if the trigonometric series

$$\sum_{n=0}^{\infty}(a_n \cos nx + b_n \sin nx)$$

converges for all values of x in any interval $a \leqq x \leqq b$, however short, then

$$\lim_{n=\infty} a_n = 0 \qquad \text{and} \qquad \lim_{n=\infty} b_n = 0.$$

More generally, the conclusion holds if merely

$$\lim_{n=\infty}(a_n \cos nx + b_n \sin nx) = 0$$

for all points of the interval.

His proof is not simple, being based on arithmetic considerations of an intricate character. The following proof has the advantage of transparency as well as of rigor, and it admits of extension to cases to which CANTOR's methods do not apply.

Let A_n and γ_n be so chosen that

$$a_n \cos nx + b_n \sin nx = A_n \sin(nx + \gamma_n),$$

where $A_n \geqq 0$. Then

$$A_n = \sqrt{a_n^2 + b_n^2}.$$

* Presented to the Society April 24, 1909.

† G. CANTOR, *Ueber einen die trigonometrischen Reihen betreffenden Lehrsatz*, Journal für Mathematik, vol. 72 (1870), p. 130; *Sur les séries trigonométriques*, Acta Mathematica, vol. 2 (1883), p. 329. For other proofs cf. HARNACK, Mathematische Annalen, vol. 19 (1881), p. 251 and p. 524; LEBESGUE, *Leçons sur les séries trigonométriques*, 1906, p. 110.

Obviously a necessary and sufficient condition that

$$\lim_{n=\infty} a_n = 0 \qquad \text{and} \qquad \lim_{n=\infty} b_n = 0$$

is that

$$\lim_{n=\infty} A_n = 0.$$

Hence it is enough for the proof of the theorem to show that *from the hypothesis*

$$\lim_{n=\infty} A_n \sin(nx + \gamma_n) = 0$$

follows that

$$\lim_{n=\infty} A_n = 0.$$

Suppose the conclusion were false. Then there would exist a positive constant h such that

$$A_n \geqq h$$

for an infinite set of values of the index, $n = n_1, n_2, \cdots$. Thus

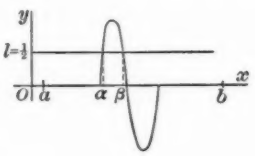
$$A_{n_i} \geqq h \qquad (i = 1, 2, \cdots).$$

The function

$$\sin(n_i x + \gamma)$$

— we will drop the index of γ_n, since the value of the latter quantity is immaterial for our proof — has the period

$$\omega_i = \frac{2\pi}{n_i}.$$



Choose $i = i_1$ so that

$$\omega_{i_1} < \tfrac{1}{2}(b - a).$$

Then the above function has a complete arch above the axis of x, the base of the arch being enclosed in the interval (a, b). Hence throughout a certain subinterval $\alpha_1 \leqq x \leqq \beta_1$, where $a \leqq \alpha_1 < \beta_1 \leqq b$,

$$\sin(n_{i_1} x + \gamma) \geqq \tfrac{1}{2}.$$

Thus we have

$$A_n \sin(nx + \gamma) \geqq \tfrac{1}{2}h$$

when

$$n = n_{i_1} \qquad \text{and} \qquad \alpha_1 \leqq x \leqq \beta_1.$$

Proceeding next to the interval (α_1, β_1) we repeat the foregoing reasoning, choosing $i = i_2 > i_1$ so that

$$\omega_{i_2} < \tfrac{1}{2}(\beta_1 - \alpha_1),$$

and thus obtaining a second arch above the axis of x, its base being included

in the interval (α_1, β_1). Thus we get an interval $\alpha_2 \leqq x \leqq \beta_2$, where $\alpha_1 \leqq \alpha_2 < \beta_2 \leqq \beta_1$, in which

$$\sin(n_{i_2}x + \gamma) \geqq \tfrac{1}{2}.$$

Hence we have

$$A_n \sin(nx + \gamma) \geqq \tfrac{1}{2}h$$

when

$$n = n_{i_2} \qquad \text{and} \qquad \alpha_2 \leqq x \leqq \beta_2.$$

A continued repetition of the process leads to an infinite sequence of intervals (α_k, β_k), each lying in its predecessor, whose lengths approach 0 as their limit and which are such that

$$A_n \sin(nx + \gamma) \geqq \tfrac{1}{2}h$$

when

$$n = n_{i_k} \qquad \text{and} \qquad \alpha_k \leqq x \leqq \beta_k.$$

The extremities of these intervals determine a point

$$\lim_{k=\infty} \alpha_k = \lim_{k=\infty} \beta_k = X$$

pertaining to all of them. For this point

$$A_n \sin(nX + \gamma) \geqq \tfrac{1}{2}h \qquad (n = n_{i_1}, n_{i_2}, \cdots).$$

This is contrary to the hypothesis that

$$\lim_{n=\infty} A_n \sin(nx + \gamma) = 0$$

for every point of the interval (a, b). Hence the theorem is true.

A more general form of CANTOR's theorem, to which both his proof and the one here given apply, is obtained by replacing the expression

$$a_n \cos nx + b_n \sin nx \qquad \text{by} \qquad a_n \cos \rho_n x + b_n \sin \rho_n x,$$

where ρ_n is any function of n which merely becomes infinite with n:

$$\lim_{n=\infty} |\rho_n| = \infty.$$

§ 2. A GENERALIZATION.

THEOREM. *Let $f_n(x)$ be a continuous function of x in the interval $a \leqq x \leqq b$; let l be a certain positive constant, and let f_n have the following property. To every subinterval of (a, b), however short, there corresponds a positive integer m such that, if $n \geqq m$ be chosen at pleasure,*

$$|f_n(x)| > l$$

at some point of the latter interval.

If under these conditions the series

$$\sum_{n=1}^{\infty} A_n f_n(x)$$

converges for every value of x in the interval (a, b), then

$$\lim_{n=\infty} A_n = 0.$$

The foregoing proof applies with slight modifications to this more general case. Suppose the theorem false. Then there exists a positive constant h such that

$$|A_{n_i}| \geqq h \qquad (i=1, 2, \cdots).$$

Determine m_1 so that, for any given $n \geqq m_1$,

$$|f_n(x)| > l$$

at some point of the interval (a, b). Denote by i_1 a value of i for which $n_i \geqq m_1$. Then there will be a subinterval $\alpha_1 \leqq x \leqq \beta_1$, where $a \leqq \alpha_1 < \beta_1 \leqq b$, in every point of which

$$|f_n(x)| \geqq l \qquad (n = n_{i_1}).$$

Hence

$$|A_n f_n(x)| \geqq hl$$

when

$$n = n_{i_1} \qquad \text{and} \qquad \alpha_1 \leqq x \leqq \beta_1.$$

A continued repetition of the process leads to a set of intervals (α_k, β_k) like those of § 1, for which

$$|A_n f_n(x)| \geqq hl$$

where

$$n = n_{i_k} \qquad \text{and} \qquad \alpha_k \leqq x \leqq \beta_k.$$

Their extremities determine a point

$$\lim_{k=\infty} \alpha_k = \lim_{k=\infty} \beta_k = X$$

pertaining to all of them. For this point

$$|A_n f_n(X)| \geqq hl \qquad (n = n_{i_1}, n_{i_2}, \cdots),$$

and thus the theorem is proven.

Series of Functions of Several Variables. The theorem admits of immediate extension to functions of several variables. Thus, if $f_n(x, y)$ is continuous throughout a region S of the (x, y)-plane, we shall demand that to every subregion Σ of S, no matter how small, there correspond a positive integer m such

that, if $n \geqq m$ be chosen at pleasure,

$$|f_n(x, y)| > l$$

at some point of Σ, l denoting as before a positive constant. It is immaterial whether S include its boundary, just as it was unimportant that the interval (a, b) include its extremities.

The proof follows precisely the same lines as in the earlier case, a succession of squares, for example,—$\Sigma_1, \Sigma_2, \cdots, \Sigma_k, \cdots$,—each lying in its predecessor, replacing the intervals (α_k, β_k).

§ 3. Applications. Zonal Harmonics. Bessel's Functions.

Zonal Harmonics. The zonal harmonics $P_n(x)$ have precisely the property of the functions $f_n(x)$ in § 2, and thus we have the

Theorem. *If a series of zonal harmonics,*

$$A_0 P_0(x) + A_1 P_1(x) + \cdots,$$

converges at all points of an interval $a \leqq x \leqq b$, however short, lying in the interval $-1 < x < 1$, then the coefficients A_n approach 0 as their limit.

Bessel's Functions. If we write

$$\sqrt{x} J_\nu(x) = \sqrt{\frac{2}{\pi}} \cos(x - \gamma) + \zeta,$$

where $\gamma = (\nu + \frac{1}{2})\pi/2$, it is well known that

$$\lim_{x=\infty} \zeta = 0.$$

We are thus enabled to state the following theorem regarding the behavior of the coefficients in a convergent series of Bessel's functions.

Theorem. *If the series*

$$A_1 J_\nu(\lambda_1 x) + A_2 J_\nu(\lambda_2 x) + \cdots,$$

where $\lambda_n > 0$ does not depend on x and $\lim_{n=\infty} \lambda_n = \infty$ converges at all points of an interval $0 < a \leqq x \leqq b$, however short, then

$$\lim_{n=\infty} \frac{A_n}{\sqrt{\lambda_n}} = 0, \qquad \text{or} \qquad |A_n| < \epsilon \sqrt{\lambda_n},$$

where ϵ is an arbitrarily small positive quantity and $n \geqq m$.

For the general term can be written in the form

$$A_n J_\nu(\lambda_n x) = \frac{A_n}{\sqrt{\lambda_n}} \sqrt{\lambda_n} J_\nu(\lambda_n x),$$

and if we set

$$f_n(x) = \sqrt{\lambda_n} J_\nu(\lambda_n x),$$

then

$$|f_n(x)| = \frac{1}{\sqrt{x}} \sqrt{\lambda_n x}\, |J_\nu(\lambda_n x)| \geqq \frac{1}{\sqrt{b}} |\sqrt{\lambda_n x} J_\nu(\lambda_n x)|.$$

Hence the series

$$\sum_{n=1}^{\infty} \left(\frac{A_n}{\sqrt{\lambda_n}} \right) f_n(x)$$

satisfies the conditions of § 2, and this proves the theorem.

We have cited the asymptotic expansion of $J_\nu(x)$; but all that we need for our proof is the fact that the continuous function $\sqrt{x} J_\nu(x)$ takes on values numerically greater than a certain positive constant l in every interval whose length exceeds a certain fixed quantity ω and which lies beyond a definite point $x = \bar{x}$. By the aid of the asymptotic expansions the theorems of this paragraph can also be established by Cantor's methods.

§ 4. Developments in Multiple Series.

The theorem of § 2 can readily be extended to developments in multiple series of functions of several variables, like

$$\sum_{m,n} A_{m,n} \sin mx \cos ny.$$

The multiple series here considered are said to converge if every simple series formed out of the terms of the multiple series converges. Thus if such a multiple series converges at all, it converges absolutely.

Let us plot the points (m, n) in a plane, and let $R_1, R_2, \cdots$ be a sequence of regions each lying in its predecessor and each containing an infinite number of points (m, n). Examples: If (ξ, η) is an arbitrary point of the plane, we may take as R_N the points (ξ, η) for which

$$(a) \qquad \xi \geqq N, \qquad \eta \geqq N;$$

or, again, those for which

$$(b) \qquad \xi + \eta \geqq N, \qquad \xi \geqq 0, \qquad \eta \geqq 0.$$

Theorem. *Let $f_{m,n}(x, y)$ be a continuous function of the two independent variables x, y in a region S of the (x, y)-plane; let l be a certain positive constant, and let f have the following property. To every subregion Σ of S, no matter how small, there corresponds a region R_N such that, if the point (m, n) be chosen at pleasure in R_N,*

$$(1) \qquad |f_{m,n}(x, y)| > l$$

at some point of Σ.

If under these conditions the series

$$\sum_{m,\,n} A_{m,\,n} f_{m,\,n}(x,\,y)$$

converges at every point $(x,\,y)$ *of* S, *then to an arbitrarily small positive number* ϵ *there corresponds a positive integer* p *such that, for every point* $(m,\,n)$ *in* R_p,

$$|A_{m,\,n}| < \epsilon.$$

If, in particular, the regions R_k *are those of Example* (b), *then*

$$\lim_{m=\infty,\,n=\infty} A_{m,\,n} = 0.$$

The proof is similar to that given in § 2. Suppose the theorem false. Then there must be a point $(m,\,n)$ in each R_k for which

$$|A_{m,\,n}| \geqq h,$$

where h denotes a certain positive constant. Thus we are led to a sequence of points $(m_i,\,n_i)$, for which

$$|A_{m_i,\,n_i}| \geqq h \qquad (i = 1,\,2,\,\cdots),$$

and such that, for any preassigned R_k, all those points for which $i \geqq q$ lie in that R_k.

Now choose R_{N_1} so that, for every point $(m,\,n)$ in R_{N_1}, relation (1) holds at some interior point of S, and let i_1 be a value of i for which $(m_{i_1},\,n_{i_1})$ lies in R_{N_1}. Then there will be an interior point $(x_1,\,y_1)$ of S, for which (1) holds when $m = m_{i_1}$ and $n = n_{i_1}$. Surround this point by a square Σ_1 lying wholly in S and such that

$$|f_{m,\,n}(x,\,y)| \geqq l \tag{2}$$

when $m = m_{i_1}$, $n = n_{i_1}$ and the point $(x,\,y)$ lies within or on the boundary of Σ_1.

The reasoning is now repeated, a smaller square Σ_2 lying in Σ_1 being obtained, throughout which relation (2) holds if $m = m_{i_2}$, $n = n_{i_2}$ and the point $(x,\,y)$ lies in Σ_2. And so on. Thus a point $(X,\,Y)$ is found, pertaining to all the squares and such that

$$|f_{m,\,n}(X,\,Y)| \geqq l$$

when $(m,\,n)$ denotes successively the points

$$(m_{i_k},\,n_{i_k}) \qquad (k = 1,\,2,\,\cdots).$$

Hence for these values of $(m,\,n)$

$$|A_{m,\,n} f_{m,\,n}(X,\,Y)| \geqq hl,$$

and the theorem is proven.

The extension of both theorem and proof to r-fold series of functions of s independent variables presents no difficulty.

By combining the above theorem and that of § 2 we are enabled to state a further theorem.

THEOREM. *If the series*

$$\sum_{m,n} A_{m,n} f_{m,n}(x, y)$$

satisfies the conditions of the foregoing theorem, the regions R_k being those of Example (a); and if, furthermore, every simple series formed out of the double series by holding one of the indices m, n fast satisfies the conditions of the theorem of § 2, the l of this latter theorem, however, being at liberty to vary from series to series, then

$$\lim_{m=\infty,\, n=\infty} A_{m,n} = 0.$$

We wish to show that, under the above hypotheses, a positive ϵ being chosen at pleasure, there exists a positive integer M such that

$$|A_{m,n}| < \epsilon$$

for all values of m and n for which

$$m + n \geqq M.$$

To do this, choose k so that for all points (m, n) of R_k

$$|A_{m,n}| < \epsilon.$$

Next, consider the series

$$\sum_{m=1}^{\infty} A_{m,n'} f_{m,n'}(x, y),$$

where $1 \leqq n' \leqq k-1$. By the theorem of § 2 its coefficients approach the limit 0, and hence

$$|A_{m,n'}| < \epsilon \qquad (m \geqq \mu').$$

Let μ be the largest of the $k-1$ numbers μ' corresponding to $n' = 1, 2, \cdots k-1$. Then

$$|A_{m,n}| < \epsilon \qquad (m \geqq \mu,\ 1 \leqq n \leqq k-1).$$

In like manner consider the series

$$\sum_{n=1}^{\infty} A_{m',n} f_{m',n}(x, y),$$

where $1 \leqq m' \leqq k-1$. Its coefficients satisfy the condition

$$|A_{m',n}| < \epsilon \qquad (n \geqq \nu').$$

Let ν be the largest of the $k-1$ numbers ν' corresponding to $m'=1, 2, \cdots k-1$. Then

$$|A_{m,n}|<\epsilon \qquad (1\leqq m\leqq k-1,\ n\geqq \nu).$$

Thus we see that

$$|A_{m,n}|<\epsilon$$

for all but a finite number of points (m, n), and hence, in particular, for all points (m, n) of a suitably chosen R_M of Example (b):

$$m+n\geqq M.$$

The conditions of the last part of the foregoing theorem are herewith seen to be fulfilled, and the present theorem is thus proven.

This theorem also admits of extension to multiple series of higher order than the second. In particular, for triple series of functions of three independent variables it can be formulated as follows.

THEOREM. *Let*

$$\sum_{m,n,p} A_{m,n,p} f_{m,n,p}(x, y, z)$$

be a triple series satisfying the conditions of the first theorem of this paragraph, the regions R_N being those of Example (a):

$$R_N: \qquad m\geqq N, \qquad n\geqq N, \qquad p\geqq N.$$

Furthermore, let every double series formed out of the triple series by holding one of the three indices m, n, p fast satisfy the conditions of the foregoing theorem. Then

$$\lim_{m=\infty,\, n=\infty,\, p=\infty} A_{m,n,p}=0.$$

§5. APPLICATION TO MULTIPLE TRIGONOMETRIC SERIES.

Consider any one of the double trigonometric series

$$\sum_{m,n} A_{m,n} \sin mx \sin ny,$$

where either or both of the sine factors may be replaced by the corresponding cosines. Let such a series converge at all points of a two dimensional region S of the (x, y)-plane. Then it can be shown exactly as in §1 that this series satisfies the conditions of the first theorem in §4, the regions R being those of Example (a).

Next, hold n fast and give y a constant value for which $\sin ny$ (or $\cos ny$) does not vanish. Thus the conditions of the theorem of §2 are seen to be ful-

filled. Hence all the conditions of the second theorem in § 4 are satisfied and we have

$$\lim_{m=\infty,\, n=\infty} A_{m,n} = 0.$$

Similarly it can be shown that if any one of the triply infinite series

$$\sum_{m,\, n,\, p} A_{m,n,p} \sin mx \sin ny \sin pz,$$

where the sines may be replaced by cosines at pleasure, converges at all points of a three dimensional region V of space, its coefficients must approach the limit zero.

HARVARD UNIVERSITY,
April, 1909.

EQUIVALENCE OF PAIRS OF BILINEAR OR QUADRATIC FORMS UNDER RATIONAL TRANSFORMATION*

BY

LEONARD EUGENE DICKSON

Introduction and Summary.

We consider pairs of bilinear or quadratic forms A and B with coefficients in a given field F. We seek necessary and sufficient conditions for the equivalence of A, B with A', B' under a linear transformation with coefficients in F. According as the determinant $|\lambda A + \mu B|$ is or is not identically zero, the case is called singular or non-singular, respectively. Both cases are treated in this paper.

In the non-singular case we may, without loss of generality (§ 3), assume that $|A| \neq 0$, $|A'| \neq 0$. By a well-known theorem, readily proved, a necessary condition for the equivalence in F of the two pairs of forms is that they shall have the same elementary divisors $(\lambda - c_i)^{e_i}$, or, if we prefer, the same rationally determined invariant factors. This condition is known to be sufficient when the forms are bilinear (§ 4), but is in general not sufficient when the forms are quadratic.

To obtain the advantage of decided simplification in the algebraic manipulation, we adjoin the roots c_i of $|\lambda A - B| = 0$ and operate in the field $F(c_1, \cdots, c_n)$. The treatment is such that the irrationalities c_i may ultimately be eliminated and the operations replaced by rational operations in the initial field F. This is accomplished by employing new variables falling into sets of conjugates with respect to F(§ 1). Of several methods of treating bilinear forms in the field of all complex numbers, that by WEIERSTRASS is best adapted to the generalization to an arbitrary field F, but with a certain modification (§ 2) to secure the desired conjugacy of variables. To accomplish this essential simplification in the nature of the new variables, we must content ourselves with a preliminary normal form (2) involving certain factors f_σ not occurring in WEIERSTRASS's unique normal form. In view of the resulting conjugacies of the new variables with respect to F, two pairs of forms A, B and A', B', having the same elementary divisors, are equivalent in F if and only if their normal forms (2) and

* Presented to the Society (Chicago) April 10, 1909.

(2′) are equivalent in $F(c_1, \cdots, c_n)$. This is always the case for bilinear forms (§ 4). For quadratic forms, the question of the equivalence of (2) and (2′) is shown to reduce to the question of the equivalence of corresponding component pairs, where a component contains only the terms having equal c_σ and equal e_σ (and hence equal elementary divisors). Two component pairs corresponding to an elementary divisor of multiplicity s (i. e., occurring s times) are equivalent if and only if the corresponding s-ary quadratic forms $\sum f_\sigma z_\sigma^2$ and $\sum f'_\sigma z_\sigma^2$, obtained by* replacing each WEIERSTRASS combination Z by the square of a single variable z, are equivalent in $F(c_\sigma)$. Hence *the question of the equivalence of pairs of quadratic forms has been reduced to that for single* † *quadratic forms.* For the field of all real numbers or for any finite field, the criteria are quite simple (§§ 12, 13) and there results a complete classification of pairs of quadratic forms in these fields.

The success of the method of treatment depends not only upon the use of conjugate variables but also upon the employment of a certain device. It appears to be impracticable to consider the system of quadratic conditions which arise when we proceed directly to determine a linear transformation T which shall replace a given pair of quadratic forms A, B by a second given pair A', B'. We make use of the following indirect procedure. By the differentiation of $A' = A$ and of the equations for T, we obtain a necessary form for T^{-1}; also a second form from $B' = B$. The conditions that the resulting two inverses of T shall be identical are linear in the unknown coefficients of T.

The same device is employed in § 14 to reduce the singular case to the non-singular case. We here make use of the rational normal type obtained by KRONECKER in his fundamental paper of 1890. This classic work of KRONECKER, exposing the very heart of the subject and applicable in complete generality to any domain of rationality, has not been in the least superseded by subsequent work, as claimed by MUTH.‡ It is not necessary to discredit the work of the master KRONECKER, writing in the spirit of general algebra, in order to honor FROBENIUS for his later elegant proof for the special case of the domain of all complex numbers.

Introduction of variables conjugate with respect to F.

1. Beginning with an example, let c_1, c_2, c_3 be the roots of a cubic equation irreducible in the field F. If x_1, x_2, x_3 are the initial variables, let

$$T: \qquad z_i = \alpha(c_i)x_1 + \beta(c_i)x_2 + \gamma(c_i)x_3 \equiv X_1 + c_i X_2 + c_i^2 X_3 \quad (i=1, 2, 3)$$

be linearly independent functions in which the coefficients of the polynomials

* Or, by replacing e_σ by unity.

† These Transactions, vol. 7 (1906), p. 275; Bulletin of the American Mathematical Society, vol. 14 (1907), p. 108.

‡ *Elementartheiler*, bottom of p. 125, top of p. 126.

$\alpha(\rho), \cdots$ belong to F. Then X_1, X_2, X_3 are independent linear functions of the x_i with coefficients in F. Let $Q(x)$ be a quadratic form, with coefficients in F, which is reducible to the form $N(z)$ by the transformation T. Let $q(y)$ be a second form reducible to $N(z)$ by the analogous transformation

$$T': \qquad z_i = \lambda(c_i)y_1 + \mu(c_i)y_2 + \nu(c_i)y_3 \equiv Y_1 + c_i Y_2 + c_i^2 Y_3 \quad (i=1, 2, 3).$$

Then $Q(x)$ is reduced to $q(y)$ by the transformation $x_i = \sum d_{ij} y_j$ obtained by solving $X_i = Y_i$ and hence with coefficients d in F.

For either T or T' the variables z_i are said to be conjugate with respect to the field F, since z_i is derived from z_1 by replacing c_1 by a root c_i of the same irreducible equation.

In general, let $x_1, \cdots, x_n$ be the initial variables. For $n = d_1 + \cdots + d_r$, let $D_s(\lambda)(s = 1, \cdots, r)$ be irreducible in F and of degree d_s; the case in which two or more D's coincide is not excluded. Let $c_{si}(i = 0, \cdots, d_s - 1)$ be the roots of $D_s = 0$. We introduce r sets of new variables, those in the sth set being

$$z_{si} = \sum_{k=1}^{n} \alpha_{sk}(c_{si})x_k \equiv \sum_{j=0}^{d_s-1} f_{sj}(x)c_{si}^j \qquad (i=0, \cdots, d_s-1),$$

in which the coefficients of the functions α and f belong to the field F. Since z_{si} is obtained from z_{s0} by replacing c_{s0} by c_{si}, the d_s new variables z_{si} are said to be conjugate with respect to F. Now the determinant

$$|c_{si}^j| \neq 0 \qquad (i, j=0, \cdots, d_s-1).$$

Hence, for each s, $f_{sj}(j = 0, \cdots, d_s - 1)$ may be introduced as new variables in place of $z_{si}(i = 0, \cdots, d_s - 1)$. Thus the r sets of conjugate variables z_{si} may be replaced by n independent variables $f_{sj}(x)$ with coefficients in F.

If in the normal forms in the z's for two bilinear or quadratic forms A and B, we replace the z_{si} by the f_{sj}, we obtain normal forms with coefficients in F which define a decomposition of A and B into component pairs each indecomposable in F (§ 11).

The modified WEIERSTRASS *normal form with conjugate variables*, §§ 2–3.

2. We make use of WEIERSTRASS's* reduction of a pair of bilinear or quadratic forms, but with a certain modification which leads to new variables falling into sets of conjugates with respect to the initial field F. The latter is any field, not having modulus 2, which contains the coefficients of the forms. While the introduction of the roots c_σ of the characteristic equation $|\lambda A - B| = 0$ is essential to the theory, this is not the case with the further irrationalities $\sqrt{C_\kappa}$. By avoiding the introduction of the latter, we shall secure

* We shall refer to the exposition by MUTH, *Elementartheiler*, pp. 69–85, 118–122, which embodies certain results by FROBENIUS essential to the completion of WEIERSTRASS's treatment.

the desired conjugacies of the new variables with respect to F. To this end we modify WEIERSTRASS's procedure (MUTH, p. 73), as follows: We set

$$S^{(\kappa-1)}S^{(\kappa)} = (\lambda - c)^{l_\kappa - 1 + l_\kappa} C_\kappa T,$$

where T is a rational integral function of $\lambda - c$ with the constant term unity. Hence T equals the square of a power series q in $\lambda - c$ with the constant term unity. Proceeding as in MUTH, p. 74, with Q replaced by q, we obtain his formula (11) with the irrational factor deleted, and his formula (12) with the additional factor $f_\kappa = C_\kappa^{-1}$ inserted. Ultimately we obtain the following result:

If A and B are bilinear forms on $2n$ variables with coefficients in the field F, and $|A| \neq 0$, and if the elementary divisors of $|\lambda A - B|$ are

$$(1) \qquad (\lambda - c_1)^{e_1}, \ \cdots, \ (\lambda - c_m)^{e_m} \qquad (e_1 + \cdots + e_m = n),$$

there exist $2n$ independent linear homogeneous functions $X_{\sigma\mu}$, $Y_{\sigma\nu}$ of the initial variables such that f_σ and the coefficients of $X_{\sigma\mu}$, $Y_{\sigma\nu}$ are rational integral functions of c_σ in F, and such that

$$(2) \quad A = \sum_{\sigma=1}^{m} f_\sigma Z_{e_\sigma}, \quad B = \sum_{\sigma=1}^{m} f_\sigma (c_\sigma Z_{e_\sigma} + Z_{e_\sigma - 1}), \quad Z_{e_\sigma} \equiv \sum_{\mu=0}^{e_\sigma - 1} X_{\sigma\mu} Y_{\sigma, e_\sigma - 1 - \mu},$$

where the second Z in B is to be deleted if $e_\sigma = 1$. If A and B are quadratic forms, $Y_{\sigma\mu} = X_{\sigma\mu}$.

If $c_1, \cdots, c_r$ are the roots of the same irreducible factor in F of $|\lambda A - B|$, then $X_{1\mu}, \cdots, X_{r\mu}$ are conjugate with respect to F; likewise $Y_{1\nu}, \cdots, Y_{r\nu}$. We have therefore secured the desired conjugacies of the new variables.

3. If we remove the restriction that $|A| \neq 0$ and assume merely that the determinant $|\lambda A + \mu B|$ is not identically zero in λ, μ, we readily obtain a pair of forms $\bar{A}$, $\bar{B}$ to which the preceding theorem may be applied. Let g and h be constants in F for which $|gA + hB| \neq 0$, g' and h' constants in F for which $gh' - g'h = 1$. Then (MUTH, p. 83),

$$\bar{A} = gA + hB, \qquad \bar{B} = g'A + h'B$$

can be reduced to the type (2). We may then deduce normal forms for A, B. Instead of the latter more complicated forms, it suffices to employ (2) in determining the conditions for the equivalence of A, B with A', B' in the field F. For, if these pairs are equivalent in F, then $\bar{A}$, $\bar{B}$ and

$$\bar{A}' = gA' + hB', \qquad \bar{B}' = g'A' + h'B'$$

are equivalent in F, and conversely.

4. For the case of bilinear forms we set

$$X_{\sigma\mu} = f_\sigma^{-1} X^*_{\sigma\mu}, \qquad Y_{\sigma\mu} = Y^*_{\sigma\mu} \quad (\sigma = 1, \cdots, m; \ \mu = 0, \cdots, e_\sigma - 1),$$

and obtain from (2) a normal form in which each $f^*_\sigma = 1$. The new variables X^*, Y^* fall into sets of conjugates with respect to F. Hence in the non-singular case, two pairs of bilinear forms with coefficients in the field F are equivalent in F if and only if their characteristic determinants have the same elementary divisors (or the same invariant factors). We have therefore obtained by a modification of WEIERSTRASS's method a proof of the theorem due to FROBENIUS.

A like argument does not hold for quadratic forms.

Ultimate normal forms, §§ 5–10.

5. Henceforth, we consider the equivalence of a pair of quadratic forms (2), for $Y = X$, with a second pair

$$(2') \quad A' = \sum_{\sigma=1}^{m} f'_\sigma Z'_{e_\sigma}, \quad B' = \sum_{\sigma=1}^{m} f'_\sigma (c_\sigma Z'_{e_\sigma} + Z'_{e_\sigma - 1}), \quad Z'_{e_\sigma} \equiv \sum_{\mu=0}^{e_\sigma - 1} X'_{\sigma\mu} X'_{\sigma\, e_\sigma - 1 - \mu},$$

having the same elementary divisors as the pair (2). Here f'_σ, like f_σ, is a non-vanishing rational integral function of c_σ with coefficients in F. The variables X are conjugate linear functions of the initial variables x_i, and the X' are conjugate functions of the x'_i. Hence (§ 1) the initial pairs A, B and A', B' are equivalent in F if and only if their normal forms (2) and (2′) are equivalent under a transformation

$$(3) \qquad X_{\sigma\mu} = \sum_{s=1}^{m} \sum_{t=0}^{e_s - 1} d^{\sigma\mu}_{st} X'_{st} \quad (\sigma = 1, \cdots m\,;\ \mu = 0, \cdots, e_\sigma - 1),$$

in which the d's are elements of $F(c_1, \cdots, c_m)$ which satisfy certain conditions imposed by the conjugacies of the variables.

We seek the conditions under which $A' = A$, $B' = B$ in view of relations (3). By differentiation with respect to X'_{st}, we get

$$(4) \qquad \begin{aligned} 2f'_s X'_{s\, e_s - 1 - t} &= \sum_{\sigma=1}^{m} f_\sigma K_{e_\sigma}, \\ 2f'_s c_s X'_{s\, e_s - 1 - t} + 2f'_s X'_{s\, e_s - 2 - t} &= \sum_{\sigma=1}^{m} f_\sigma (c_\sigma K_{e_\sigma} + K_{e_\sigma - 1}), \end{aligned}$$

where the second X' in (4_2) is to be suppressed if $t = e_s - 1$, while

$$K_{e_\sigma} \equiv \sum_{\mu=0}^{e_\sigma - 1} X_{\sigma\mu} d^{\sigma\mu'}_{st} + \sum_{\mu=0}^{e_\sigma - 1} X_{\sigma\mu'} d^{\sigma\mu}_{st} \qquad (\mu' = e_\sigma - 1 - \mu).$$

Since μ and μ' range over the same values, the sums are equal and

$$(5) \qquad K_{e_\sigma} = 2 \sum_{\mu=0}^{e_\sigma - 1} d^{\sigma\mu}_{st} X_{\sigma\, e_\sigma - 1 - \mu}.$$

Hence (4_1) and (4_2) must define the same transformation, viz., the inverse of (3). Conversely,* if (4_1) and (4_2) define the same transformation inverse to (3) and if $|d| \neq 0$, then the pair (2) is equivalent to the pair (2′) under the transformation (3).

6. Replacing e_σ by $e_\sigma - 1$ and μ by $\mu - 1$ in (5), we get

$$K_{e_\sigma - 1} = 2 \sum_{\mu=1}^{e_\sigma - 1} d_{st}^{\sigma\,\mu-1} X_{\sigma\, e_\sigma - 1 - \mu}. \tag{6}$$

We compare the relations (4) with the values (5) and (6) inserted. For $t = e_s - 1$, (4_2) must equal the product of (4_1) by c_s, whence

$$(c_s - c_\sigma) d_{s\,e_s-1}^{\sigma\,0} = 0, \qquad (c_s - c_\sigma) d_{s\,e_s-1}^{\sigma\,\mu} = d_{s\,e_s-1}^{\sigma\,\mu-1} \quad (\sigma = 1, \cdots, m;\ \mu = 1, \cdots, e_\sigma - 1).$$

It follows readily that

$$d_{s\,e_s-1}^{\sigma\,\mu} = 0 \ (0 \leqq \mu \leqq e_\sigma - 1) \text{ for } c_\sigma \neq c_s; \quad d_{s\,e_s-1}^{\sigma\,\nu} = 0 \ (0 \leqq \nu \leqq e_\sigma - 2) \text{ for } c_\sigma = c_s. \tag{7}$$

For $0 \leqq t \leqq e_s - 2$, we substitute into (4_2) the expression (4_1) for X' and that derived from it by replacing t by $t + 1$. We obtain the conditions

$$(c_\sigma - c_s) d_{st}^{\sigma\,0} = d_{s\,t+1}^{\sigma\,0}, \qquad (c_\sigma - c_s) d_{st}^{\sigma\,\mu} = d_{s\,t+1}^{\sigma\,\mu} - d_{st}^{\sigma\,\mu-1} \quad (\mu = 1, \cdots, e_\sigma - 1). \tag{8}$$

First, let σ and s be such that $c_\sigma \neq c_s$. For $\mu = 0$ in (7_1) and $t = e_s - 2$ in (8_1), we get $d_{s\,e_s-2}^{\sigma\,0} = 0$. Then for $t = e_s - 3$ in (8_1), we get $d_{s\,e_s-3}^{\sigma\,0} = 0$. Proceeding similarly with (8_1), we establish (9) for $i = 0$:

$$d_{sk}^{\sigma\,i} = 0 \quad (0 \leqq i \leqq e_\sigma - 1,\ 0 \leqq k \leqq e_s - 1,\ c_\sigma \neq c_s). \tag{9}$$

To proceed by induction, let (9) hold for $i = \mu - 1$. Then the final term in (8_2) is zero. Take $t = e_s - 2$ in (7_1); we get $d_{s\,e_s-2}^{\sigma\,\mu} = 0$. For $t = e_s - 3$ in (8_2), we get $d_{s\,e_s-3}^{\sigma\,\mu} = 0$. Proceeding similarly with (8_2), we find that (9) holds for $i = \mu$. Hence the induction is complete.

Next, let $c_\sigma = c_s$. Then (7_2) and (8) are satisfied if and only if

$$d_{sk}^{\sigma\,i} = 0 \text{ for } k > i, \quad d_{sk}^{\sigma\,i} = d_{s0}^{\sigma\,i-k} \text{ for } k \leqq i \quad (0 \leqq i \leqq e_\sigma - 1,\ 0 \leqq k \leqq e_s - 1,\ c_\sigma = c_s). \tag{10}$$

7. If we separate the variables X into sets, including in a single set all the $X_{\sigma\mu}$ for which the c_σ are equal, we see from (9) that the variables of any set are transformed amongst themselves by (3). We may therefore limit attention to the separate sets. Consider the set for which

$$c_1 = c_2 = \cdots = c_g, \qquad c_\gamma \neq c_1 (\gamma > g). \tag{11}$$

* For a formal verification, multiply (4_1) X'_{st} and sum for $s = 1, \cdots, m$; $t = 0, \cdots, e_s - 1$. Applying (2_3) on the left, and (3) and (2_3) on the right, we get $A' = A$. Proceeding similarly with (4_2), we get $B' = B$.

By (9) and (10), the corresponding partial transformation (3) is

$$X_{\sigma\mu} = \sum_{s=1}^{g} \sum_{t=0}^{l_s} d_{s0}^{\sigma\,\mu-t} X'_{st} \quad (\sigma=1,\cdots,g;\ \mu=0,\cdots,e_\sigma-1), \tag{12}$$

where l_s denotes the lesser of the integers μ, e_s-1. By (4_1) and (5),

$$f'_s X'_{s\,e_s-1-t} = \sum_{\sigma=1}^{g} f_\sigma \sum_{\mu=t}^{e_\sigma-1} d_{s0}^{\sigma\,\mu-t} X_{\sigma\,e_\sigma-1-\mu} \quad (s=1,\cdots,g;\ t=0,\cdots,e_s-1).$$

By a change of indices, the latter becomes

$$f'_s X'_{st} = \sum_{\sigma=1}^{g} f_\sigma \sum_{\mu=0}^{t+e_\sigma-e_s} d_{s0}^{\sigma\,e_\sigma-e_s+t-\mu} X_{\sigma\mu} \quad (s=1,\cdots,g;\ t=0,\cdots,e_s-1). \tag{13}$$

The partial transformation (12) must replace the terms of (2) with $\sigma \leqq g$ by the terms of (2′) with $\sigma \leqq g$. Subtracting c_1 times (2_1) from (2_2), and c_1 times $(2'_1)$ from $(2'_2)$, we find by (11) that

$$\sum_{\sigma=1}^{g} f_\sigma Z_{e_\sigma} = \sum_{\sigma=1}^{g} f'_\sigma Z'_{e_\sigma}, \qquad \sum_{\sigma=1}^{g} f_\sigma Z_{e_\sigma-1} = \sum_{\sigma=1}^{n} f'_\sigma Z'_{e_\sigma-1} \tag{14}$$

under transformation (12). The necessary and sufficient condition for the equivalence of the pairs (14) under (12) is that transformation (13) be inverse to (12) (see end of § 5). Since (14) are unaltered by the interchange of the accented and unaccented letters and since (13) must replace the second pair by the first, it follows that (13) must be of the same type as (12), the most general transformation replacing the first pair by the second. Hence

$$d_{s0}^{\sigma e_\sigma-e_s+t-\mu} = 0 \quad (\mu=t+1,\cdots,\ t+e_\sigma-e_s,\ \text{if } e_\sigma>e_s), \tag{15}$$

$$d_{s0}^{\sigma\,k} = 0 \quad (k=0,\cdots,\ e_\sigma-e_s-1,\ \text{if } e_\sigma>e_s). \tag{15′}$$

These equivalent formulæ are fundamental in what follows.

8. We proceed to determine and discuss the conditions under which (12) and (13) are inverse transformations. Eliminating * $X_{\sigma\mu}$ and changing † the order of the summations, we get

$$f'_s X'_{st} = \sum_{s_1=1}^{g} \sum_{t_1=0}^{\tau} \phi_{s_1t_1} X'_{s_1t_1}, \qquad \phi_{s_1t_1} = \sum_{\sigma=1}^{g} \sum_{\mu=t_1}^{\tau+e_\sigma-e_s} f_\sigma d_{s0}^{\sigma e_\sigma-e_s+t-\mu} d_{s_1 0}^{\sigma\mu-t_1}, \tag{16}$$

where τ is the lesser of t, $e_{s_1}-1$. Hence $\phi=0$ except for $s_1=s$, $t_1=t$.

We employ the abbreviation (s) to denote the set of all integers s corresponding to equal values of e_s, so that s_1 is in (s) if and only if $e_{s_1}=e_s$.

We may give a very simple expression for ϕ_{s_1t} when s_1 is in the set (s).

* In (13), $\mu \leqq t+e_\sigma-e_s \leqq e_\sigma-1$, so that each $X_{\sigma\mu}$ may be replaced by (12).

† By (15), the maximum μ in (13) is t.

Consider first the values of σ for which $e_\sigma > e_s$. Then, in (16_2) with $t_1 = t$, the first d vanishes by (15) when $\mu > t$, while the second d vanishes by (15′) when $\mu = t$. For $e_\sigma < e_s$, there occur no terms in the sum. Hence we may restrict σ to the set (s), so that

$$\phi_{s_1 t} \equiv \sum_{\sigma \text{ in } (s)} f_\sigma d_{s0}^{\sigma 0} d_{s_1 0}^{\sigma 0} = \begin{cases} f'_s \text{ if } s_1 = s \\ 0 \text{ if } s_1 \neq s \end{cases} \qquad [s_1 \text{ in } (s)]. \tag{17}$$

Now (17) are the necessary and sufficient conditions that

$$z_\sigma = \sum_{s_1 \text{ in } (s)} d_{s_1 0}^{\sigma 0} z'_{s_1} \qquad [\sigma \text{ in } (s)] \tag{18}$$

shall replace

$$\sum f_\sigma z_\sigma^2 \quad \text{by} \quad \sum f'_\sigma z'^2_\sigma \qquad [\sigma \text{ in } (s)]. \tag{19}$$

Since $\Pi f'_\sigma = |d|^2 \Pi f_\sigma$ and each $f'_\sigma \neq 0$, $f_\sigma \neq 0$, we have*

$$|d_{s_1 0}^{\sigma 0}| \neq 0 \qquad [\sigma, s_1 \text{ in } (s)]. \tag{20}$$

For the existence in $F(c_1)$ of elements $d_{s0}^{\sigma k}$ making each $\phi_{s_1 t_1} = 0$ or f'_s, a necessary condition was seen to be the existence of elements

$$d_{s0}^{\sigma 0} \qquad [\sigma \text{ in } (s);\ s = 1, \cdots, g] \tag{21}$$

which satisfy relations (17). This condition is also sufficient. In proof, we assign the value zero to every d not in the set (21). Then $\phi_{s_1 t_1}$ vanishes unless σ and s_1 are in (s) and $t = \mu = t_1$, while in that case ϕ becomes (17). Hence (12) and (13) are then inverse, a fact also evident from their special forms:

$$X_{\sigma\mu} = \sum_{s \text{ in } (\sigma)} d_{s0}^{\sigma 0} X'_{s\mu}, \qquad f'_s X'_{st} = \sum_{\sigma \text{ in } (s)} f_\sigma d_{s0}^{\sigma 0} X_{\sigma t}. \tag{22}$$

For each set (s), relations (17) express the equivalence of the quadratic forms (19) under a transformation in $F(c_1)$ of non-vanishing determinant.

9. The discussion in §§ 5–8 leads to the following

THEOREM. *Let the elementary divisors $(\lambda - c_i)^{e_i}$ be separated into sets by listing together all the equal ones (with equal c's and equal e's). Let the first set be given by $i = 1, \cdots, \gamma_1$; the second by $i = \gamma_1 + 1, \cdots, \gamma_1 + \gamma_2$; the third by $\gamma_1 + \gamma_2 + 1, \cdots, \gamma_1 + \gamma_2 + \gamma_3$, etc. Then the necessary and sufficient condition for the equivalence in the field $F(c_1, \cdots, c_m)$ of the two pairs of*

* Another proof follows from (12) for $\mu = 0$ and (15′) for $k = 0$. Let e_{σ_1} be the greatest e, e_{σ_2} the next to the greatest e, etc. Then

$$X_{\sigma_1 0} = \sum_{s \text{ in } (\sigma_1)} d_{s0}^{\sigma_1 0} X'_{s0}, \qquad X_{\sigma_2 0} = \sum_{s \text{ in } (\sigma_1)} d_{s0}^{\sigma_2 0} X'_{s0} + \sum_{s \text{ in } (\sigma_2)} d_{s0}^{\sigma_2 0} X'_{s0}, \ \cdots.$$

Hence the determinant of (12) contains the non-vanishing factors

$$|d_{s0}^{\sigma_1 0}|, \ s \text{ and } \sigma_1 \text{ in } (\sigma_1); \qquad |d_{s0}^{\sigma_2 0}|, \ s \text{ and } \sigma_2 \text{ in } (\sigma_2); \ \cdots.$$

quadratic forms (2) *and* (2′) *is the equivalence in* $F(c_\sigma)$ *of each individual quadratic form*

$$(23) \qquad \sum_{\sigma=1}^{\gamma_1} f_\sigma z_\sigma^2, \qquad \sum_{\sigma=\gamma_1+1}^{\gamma_1+\gamma_2} f_\sigma z_\sigma^2, \qquad \sum_{\sigma=\gamma_1+\gamma_2+1}^{\gamma_1+\gamma_2+\gamma_3} f_\sigma z_\sigma^2, \cdots$$

with the corresponding form $\sum f'_\sigma z_\sigma^2$. *The number of forms* (23) *is the number of distinct elementary divisors.*

If $c_1, \cdots, c_r$ are the roots of an irreducible factor in F of the characteristic function $|\lambda A - B|$, the relation between the elementary divisors and the rational invariant factors shows that the elementary divisors which correspond to $\lambda - c_i$ $(i = 2, \cdots, r)$ may be derived from those which correspond to $\lambda - c_1$ by replacing c_1 by c_i. Hence the equivalence in $F(c_1)$ of those of the forms (23) which relate to c_1 with the corresponding forms (23′) implies the equivalence in $F(c_i)$ of the forms relating to c_i.

10. Another convenient statement of the theorem of § 9 is as follows. The equivalence of the pairs (2) and (2′) requires the equivalence of the component pairs obtained by limiting the summation index σ to values giving equal c_σ's. If one set of equal c's is given by (11), the pairs (14) must be equivalent. The latter requires the equivalence of the component pairs obtained by the further limitation of σ to a set (σ) of values giving equal e's. Hence for each set (σ),

$$(24) \qquad \sum_{(\sigma)} f_\sigma Z_{e_\sigma}, \qquad \sum_{(\sigma)} f_\sigma Z_{e_\sigma - 1}$$

must be equivalent to the corresponding forms (24′) in f'_σ. Here σ ranges over the integers (σ) for which the c_σ are equal and the e_σ are equal. *Thus the pairs* (2) *and* (2′) *are equivalent in* $F(c_1, \cdots, c_m)$ *if and only if the component pairs* (24) *and* (24′) *are equivalent in* $F(c_\sigma)$. *The latter pairs are equivalent if and only if the individual forms* $\sum_{(\sigma)} f_\sigma z_\sigma^2$ *and* $\sum_{(\sigma)} f'_\sigma z_\sigma^2$ *are equivalent in* $F(c_\sigma)$. It thus suffices to replace * the combinations

$$(25) \qquad Z_{e_\sigma} \equiv \sum_{\mu=0}^{e_\sigma - 1} X_{\sigma\mu} X_{\sigma\, e_\sigma - 1 - \mu}$$

by the squares $z_{e_\sigma}^2$ of single new variables.

It will prove instructive to have a direct verification of the sufficiency of these conditions. By hypothesis the forms (19) are equivalent, so that relations (17)

*Or, to replace e_σ by unity.

hold. Now transformation (22_1) replaces the form (24_1) by

$$\sum_{\mu=0}^{e_\sigma-1} \sum_{s, s_1 \text{ in } (\sigma)} \Big(\sum_{(\sigma)} f_\sigma d_{s0}^{\sigma 0} d_{s_1 0}^{\sigma 0}\Big) X'_{s\mu} X'_{s_1 e_\sigma - 1 - \mu}.$$

By (17), the quantity in parenthesis equals f'_s or 0 according as $s_1 = s$ or $s_1 \neq s$. In view of (25) and $e_s = e_\sigma$, we obtain $\sum_{s \text{ in } (\sigma)} f'_s Z'_{e_s}$. Similarly, (24_2) is transformed into $(24'_2)$.

Decomposition of a pair of quadratic forms in F.

11. Let $c_1, \cdots, c_r$ be the distinct roots of an irreducible equation $\phi = 0$ in F. Of the elementary divisors which are powers of $\lambda - c_1$, consider a particular one $(\lambda - c_1)^{e_1}$ and the related variables

$$X_{1\mu} = V_{0\mu} + V_{1\mu} c_1 + \cdots + V_{r-1\,\mu} c_1^{r-1} \quad (\mu = 0, \cdots, e_1 - 1),$$

where the V's are linear functions of the initial variables with coefficients in F. If we replace c_1 by c_σ $(1 \leqq \sigma \leqq r)$, we obtain (end of § 9) an elementary divisor $(\lambda - c_\sigma)^{e_1}$ and related variables $X_{\sigma\mu}$. Since $|c_\sigma^k| \neq 0$, the $V_{i\mu}$ may be expressed in terms of $X_{1\mu}, \cdots, X_{r\mu}$. Hence the re_1 functions $V_{i\mu}$ are linearly independent. Thus

$$\alpha(V) = \sum_{\sigma=1}^{r} f_\sigma Z_{e_\sigma}, \qquad \beta(V) = \sum_{\sigma=1}^{r} f_\sigma (c_\sigma Z_{e_\sigma} + Z_{e_\sigma - 1})$$

are quadratic forms on the independent variables $V_{i\mu}$ with coefficients in F.

The pair α, β has the single invariant factor ϕ^{e_1}, and the elementary divisors $(\lambda - c_i)^{e_1}$ $(i = 1, \cdots, r)$. But if $\alpha = \alpha' + \alpha''$, $\beta = \beta' + \beta''$, the variables in α', β' being distinct from those in α'', β'', the elementary divisors of α, β are those of α', β', together with those of α'', β''. Hence there is no such decomposition of α, β within the field F.

THEOREM. *If $|A| \neq 0$ and $|\lambda A - B|$ has the invariant factors*

$$\phi_{11}^{e_{11}} \phi_{12}^{e_{12}} \phi_{13}^{e_{13}} \cdots, \quad \phi_{21}^{e_{21}} \phi_{22}^{e_{22}} \phi_{23}^{e_{23}} \cdots, \quad \cdots \tag{26}$$

in which each ϕ_{ij} is irreducible in F and each $e_{ij} > 0$, the pair of quadratic forms A, B decomposes in F into $\sum A_{ij}$, $\sum B_{ij}$, where each component pair A_{ij}, B_{ij} is not decomposable in F and has the single invariant factor $\phi_{ij}^{e_{ij}}$.

Field of reals; finite fields.

12. Let F be the field R of all real numbers. Consider first a pair of conjugate imaginary roots c_1 and c_2, whence $e_1 = e_2$. In the field $C = F(c_1)$ of all complex numbers, $2f_1$ is a square g_1^2 and $2f_2$ is the conjugate imaginary g_2^2. Set $X'_{1\mu} = g_1 X_{1\mu}$, $X'_{2\mu} = g_2 X_{2\mu}$. Hence we may set $f_1 = f_2 = \frac{1}{2}$ and yet preserve the conjugacy of the variables. With this preliminary normalization accomplished, we set

$$X_{1\mu} = x_{1\mu} + iy_{1\mu}, \qquad X_{2\mu} = x_{1\mu} - iy_{1\mu} \qquad (\mu = 0, \cdots, e_1 - 1),$$

$$c_1 = \gamma_1 + i\delta_1, \qquad e_2 = \gamma_1 - i\delta_1.$$

Then, by (25), $Z_{e_1} = L_{e_1} + iM_{e_1}$, where

$$(27) \qquad L_{e_1} = \sum_{\mu=0}^{e_1-1} (x_{1\mu}x_{1e_1-1-\mu} - y_{1\mu}y_{1e_1-1-\mu}), \qquad M_{e_1} = 2\sum_{\mu=0}^{e_1-1} x_{1\mu}y_{1e_1-1-\mu}.$$

Let A_{12} and B_{12} denote the terms of (2) given by $\sigma = 1, 2$. Since

$$\tfrac{1}{2}(Z_{e_1} + Z_{e_2}) = L_{e_1}, \qquad \tfrac{1}{2}(c_1 Z_{e_1} + c_2 Z_{e_2}) = \gamma_1 L_{e_1} - \delta_1 M_{e_1},$$

we have

$$(28) \qquad A_{12} = L_{e_1}, \qquad B_{12} = \gamma_1 L_{e_1} - \delta_1 M_{e_1} + L_{e_1-1}.$$

When the variables are taken in the order $x_{10}, y_{10}, x_{11}, y_{11}, \cdots$, we have

$$(29) \qquad (\lambda A_{12} - B_{12}) \equiv \begin{vmatrix} O & O & O & \cdots & O & O & P & Q \\ O & O & O & \cdots & O & P & Q & O \\ O & O & O & \cdots & P & Q & O & O \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ P & Q & O & \cdots & O & O & O & O \\ Q & O & O & \cdots & O & O & O & O \end{vmatrix}$$

where

$$(30) \qquad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad Q = \begin{pmatrix} \lambda - \gamma_1 & \delta_1 \\ \delta_1 & -\lambda + \gamma_1 \end{pmatrix}.$$

The determinant of (29) has the single invariant factor $|Q|^{e_1}$.

Next, let c_s be real and, among the elementary divisors which are powers of $\lambda - c_s$, let those exponents e be equal for which σ ranges over the set (s). Since the corresponding f_σ and f'_σ are real, the two forms (19) are equivalent in R if and only if the number of positive f_σ equals the number of positive f'_σ, where σ ranges over (s).

THEOREM. *The necessary and sufficient conditions for the equivalence under*

real linear transformation of two non-singular pairs of real quadratic forms are that they have the same elementary divisors and the same number of positive terms in the series $f_{\sigma_1}, \cdots, f_{\sigma_s}$ *and* $f'_{\sigma_1}, \cdots, f'_{\sigma_s}$ *corresponding to each set of equal real elementary divisors* $(\lambda - c_{\sigma_1})^{e_{\sigma_1}} \equiv \cdots \equiv (\lambda - c_{\sigma_s})^{e_{\sigma_s}}$.

As a real normal form for a pair of real quadratic forms we may employ two aggregates of forms affecting different sets of variables. Corresponding to each pair of conjugate imaginary elementary divisors we take as the component forms a pair of type (28). Corresponding to each real elementary divisor of multiplicity s, we take as the component forms sums like (2), with σ ranging over s values, with the value $+1$ assigned to s' of the f_σ, $0 \leqq s' \leqq s$, and the value -1 assigned to the remaining $s - s'$ of the f_σ.

13. Let the field F be the $GF[p^l]$, $p > 2$. If c_s is the root of an equation of degree r irreducible in the $GF[p^l]$, the forms (19) are equivalent in $F(c_s)$, viz., in the $GF[p^{lr}]$, if and only if $\Pi f'_\sigma \div \Pi f_\sigma$ is a square in the $GF[p^{lr}]$.

To obtain normal forms for the non-equivalent pairs of quadratic forms in the $GF[p^l]$ we may therefore employ (2), with the f_σ chosen so that, corresponding to an elementary divisor of multiplicity s, $s-1$ of the f_σ equal unity and the remaining one of the f_σ is unity or a particular not-square in the $GF[p^{lr}]$.

Reduction of the singular case to the non-singular case.

14. KRONECKER* has established the following theorem: Within any field† F not having modulus 2, a pair of quadratic forms ϕ, ψ, for which $|u\phi + v\psi| \equiv 0$, can be transformed into a pair

$$(31) \qquad A = \sum_{g,h}^{1,\cdots,M} A_{gh} X_g X_h + \sum_{q=1}^{L} \sum_{p=1}^{M_q} X^{(q)}_{p-1} X^{(q)}_{p+M_q},$$

$$B = \sum B_{gh} X_g X_h + \sum\sum X^{(q)}_p X^{(q)}_{p+M_q},$$

in which $|uA_{gh} + vB_{gh}| \not\equiv 0$.

Let $\alpha, \beta, \gamma, \delta$ be elements of F such that

$$|\alpha A_{gh} + \beta B_{gh}| \neq 0 \qquad (\alpha\delta - \beta\gamma \neq 0).$$

By a transformation on the variables $X^{(g)}$ alone, the pair

$$\alpha A + \beta B, \ \gamma A + \delta B$$

can be transformed into a pair A^*, B^*, of type‡ (31) with

*Berliner Sitzungsberichte, 1890, pp. 1375–1388. See p. 1388.

†KRONECKER limits the statement of the theorem to a domain of rationality.

‡KRONECKER, loc. cit., bottom of p. 1382.

$$A^*_{gh} = \alpha A_{gh} + \beta B_{gh}, \qquad B^*_{gh} = \gamma A_{gh} + \delta B_{gh}.$$

We may therefore restrict attention to pairs (31) with $|A_{gh}| \neq 0$.

We investigate the equivalence of a pair (31) with a like pair

$$(31') \quad A' = \sum A'_{gh} Y_g Y_h + \sum\sum Y^{(q)}_{p-1} Y^{(q)}_{\mu}, \qquad B' = \sum B'_{gh} Y_g Y_h + \sum\sum Y^{(q)}_{p} Y^{(q)}_{\mu} \quad (\mu \equiv p + M_q)$$

under a linear transformation, with coefficients in F,

$$(32) \quad \begin{aligned} X_h &= \sum_{s=1}^{M} c_{hs} Y_s + \sum_{q=1}^{L} \sum_{j=0}^{2M_q} d_{hqj} Y^{(q)}_j \qquad (h = 1, \cdots, M), \\ X^{(t)}_i &= \sum_{s=1}^{M} e_{tis} Y_s + \sum_{q=1}^{L} \sum_{j=0}^{2M_q} f_{tiqj} Y^{(q)}_j \qquad (t = 1, \cdots, L;\ i = 0, \cdots, 2M_t). \end{aligned}$$

Differentiating $A' = A$ and $B' = B$ partially with respect to $Y^{(t)}_{\lambda}$ and applying (32), we get

$$(33) \quad \begin{aligned} &2 \sum_{g,h=1}^{M} A_{gh} X_g d_{ht\lambda} \\ &\quad + \sum_{q=1}^{L} \sum_{p=1}^{M_q} (X^{(q)}_{p-1} f_{g\mu t\lambda} + X^{(q)}_{\mu} f_{qp-1\,t\lambda}) \end{aligned} = \begin{cases} Y^{(t)}_{\lambda+1+M_t} & (\lambda = 0, \cdots, M_t - 1), \\ 0 & (\lambda = M_t), \\ Y^{(t)}_{\lambda-1-M_t} & (\lambda = 1 + M_t, \cdots, 2M_t); \end{cases}$$

$$(34) \quad \begin{aligned} &2 \sum_{g,h} B_{gh} X_g d_{ht\lambda} \\ &\quad + \sum\sum (X^{(q)}_{p} f_{q\mu t\lambda} + X^{(q)}_{\mu} f_{qpt\lambda}) \end{aligned} = \begin{cases} 0 & (\lambda = 0), \\ Y^{(t)}_{\lambda+M_t} & (\lambda = 1, \cdots, M_t), \\ Y^{(t)}_{\lambda-M_t} & (\lambda = 1 + M_t, \cdots, 2M_t). \end{cases}$$

In (34_2) we replace λ by $\lambda + 1$ and compare the result with (33_1); by the coefficients of X_g, we get

$$(35) \quad \sum_{h=1}^{M} B_{gh} d_{ht\lambda+1} = \sum_{h=1}^{M} A_{gh} d_{ht\lambda} \qquad (\lambda = 0, \cdots, M_t - 1;\ g = 1, \cdots, M).$$

By the coefficient of X_g in (33_2), we get

$$\sum_{h=1}^{M} A_{gh} d_{htM_t} = 0 \qquad (g = 1, \cdots, M).$$

Since $|A_{gh}| \neq 0$, each $d_{htM_t} = 0$. Then, by (35) for $\lambda = M_t - 1$, $d_{ht\lambda} = 0$. Similarly, by taking in turn $\lambda = M_t - 2, \cdots, \lambda = 0$, we get

$$(36) \quad d_{htj} = 0 \qquad (j = 0, \cdots, M_t;\ h = 1, \cdots, M;\ t = 1, \cdots, L).$$

Thus (36) are among the necessary conditions for the identity of corresponding

Y's in (33) and (34). When the latter conditions are satisfied, we have unique expressions for the Y's in terms of the X's, and hence the form of the inverse of (32). By (34_2), $Y_k^{(t)}\,(k = 1 + M_t, \cdots, 2M_t)$ does not contain an X_g, in view of (36). Since the pairs (31) and (31′) are of like type, the inverse (32) must have the analogous property that $X_k^{(t)}\,(k = 1 + M_t, \cdots, 2M_t)$ shall not contain a variable Y_g. But every term of the double sums in (31) contains such an $X_k^{(t)}$. Hence the first sums in (31) must be transformable into the the first sums in (31′) by a transformation on the X_g alone. We may therefore state the

THEOREM. *The necessary and sufficient condition for the equivalence in F of the singular pairs* (31) *and* (31′) *is the equivalence in F of the non-singular components*

$$\sum A_{gh} X_g X_h, \qquad \sum B_{gh} X_g X_h \quad \textit{and} \quad \sum A'_{gh} Y_g Y_h, \qquad \sum B'_{gh} Y_g Y_h.$$

THE UNIVERSITY OF CHICAGO,
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ON A COMPLETE SYSTEM OF INVARIANTS OF TWO TRIANGLES*

BY

DAVID D. LEIB

Introductory.

The simpler invariants of two triangles, arising from the connex set up by them when regarded as two cubic curves of the third order and of the third class, respectively, were discussed by Dr. HUN in vol. 5 of these Transactions, pages 39–55. The present paper exhibits *a complete system* by means of which all invariant relations can be conveniently expressed, and is largely the result of an attempt to solve a problem proposed by Professor MORLEY: What is the invariant relation between two triangles so that a conic may be drawn on the vertices of one triangle and touching the sides of the other? In this and similar problems, the one triangle was invariably taken as the reference triangle, and the invariant relation consequently expressed in terms of the coefficients of the other, a simplification which we shall use throughout the article. In this way the selection of the fundamental system was largely a matter of experiment in the first place, the object being to select such as would enable one to write down invariant relations in the most simple form, and at the same time to have these fundamental invariants easily interpreted geometrically. In the body of this paper we shall follow the logical order, however, of giving the system first and then proving its completeness.

§ 1. *The fundamental system.*

If we take one triangle as a 3-line given by the symbolic equation

$$(\alpha x)(\beta x)(\gamma x) = 0,$$

and the other as a 3-point

$$(a\xi)(b\xi)(c\xi) = 0,$$

where $(\alpha x) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$, etc., the constants a_i, b_i, c_i, α_i, β_i, γ_i form the domain of rationality for the invariants. The six fundamental invariants Δ_1, D_1, I_1, D_2, I_2 and I_3, expressed rationally and integrally in this domain are given on page 128 of *The Johns Hopkins University Circular*, July, 1908. Of these, Δ_1 and D_1 are the determinants of the 3-point and 3-line respectively.

* Presented to the Society, December 31, 1908.

That is

$$\Delta_1 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}.$$

Following Dr. HUN's notation for indicating the collective degree in Roman and Greek letters respectively, the invariants are represented as follows: $D_1(3, 0)$; $\Delta_1(0, 3)$; $I_1(3, 3)$; $I_2(6, 6)$; $D_2(6, 6)$; $I_3(9, 9)$. From these can be formed four independent absolute invariants of zero degree,

$$\frac{I_2}{D_2}, \qquad \frac{I_1}{\Delta_1 D_1}, \qquad \frac{I_1^2}{I_2}, \qquad \frac{I_1 I_2}{I_3},$$

and in terms of these four can be expressed rationally every other rational invariant of zero degree in both sets of coefficients. This is a minimum reduced system, since two triangles have projectively four absolute invariants.

If now the 3-line be taken as the reference triangle, Δ_1 assumes the value unity and the other fundamental invariants become homogeneous, rational integral functions of the coefficients of the 3-point, as follows:

$$D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

$$I_1 = a_1 b_2 c_3 + a_3 b_1 c_2 + a_2 b_3 c_1 + a_1 b_3 c_2 + a_2 b_1 c_3 + a_3 b_2 c_1,$$

$$D_2 = \begin{vmatrix} a_2 a_3 & b_2 b_3 & c_2 c_3 \\ a_3 a_1 & b_3 b_1 & c_3 c_1 \\ a_1 a_2 & b_1 b_2 & c_1 c_2 \end{vmatrix},$$

$$I_2 = a_2 a_3 b_3 b_1 c_1 c_2 + a_1 a_2 b_2 b_3 c_3 c_1 + a_3 a_1 b_1 b_2 c_2 c_3 + a_2 a_3 b_1 b_2 c_3 c_1 + a_1 a_2 b_3 b_1 c_2 c_3 + a_3 a_1 b_2 b_3 c_1 c_2,$$

$$I_3 = a_1 a_2 a_3 b_1 b_2 b_3 c_1 c_2 c_3.$$

It is in this simplified form that the invariants are most available, and it is a matter of very little difficulty to introduce Δ_1 into any invariant form so as to render it homogeneous and of the proper degree in the Greek letters. The invariants D_1 and D_2 are skew, as is evident from their form.

§ 2. *Proof of the completeness of the system.*

Take the 3-line as reference triangle and call it T_1. The 3-point, $(a\xi)(b\xi)(c\xi) = 0$, we shall call T_2. If T_1 is taken as $(\alpha x)(\beta x)(\gamma x) = 0$, we shall define a rational simultaneous invariant of T_1 and T_2 as a rational integral func-

tion of the coefficients, which has the invariant property and is unaltered by any permutation of the three lines or of the three points; that is, by interchanges of α_i, β_i, γ_i or of a_i, b_i, c_i. *When T_1 is taken as the reference triangle, all rational simultaneous invariants (except Δ_1, which becomes unity) of T_1 and T_2 are homogeneous, rational, integral functions of a_i, b_i, c_i, which are unaltered by all permutations of the letters and of the subscripts.*

The converse is equally true, and any homogeneous, rational, integral function of a_i, b_i, c_i, which is left unaltered by permutation of the letters or of the subscripts, is equivalent to an explicit rational integral function homogeneous in both the Roman and the Greek letters, which has the invariant property under all linear transformations of the plane. Hence the most general form of such an invariant is

$$I = \sum a_1^{i_1} a_2^{i_2} a_3^{i_3} b_1^{j_1} b_2^{j_2} b_3^{j_3} c_1^{k_1} c_2^{k_2} c_3^{k_3},$$

where

$$i_1 + i_2 + i_3 = n, \qquad i_1 + j_1 + k_1 = n,$$

$$j_1 + j_2 + j_3 = n, \qquad i_2 + j_2 + k_2 = n,$$

$$k_1 + k_2 + k_3 = n, \qquad i_3 + j_3 + k_3 = n.$$

We next prove the lemma: *Every individual term of I, where I is of the nth degree in the coefficients of the 3-point, is the product of n factors of the type $a_i b_k c_l$ ($i \neq k \neq l$, $i = 1, 2, 3$; $k = 1, 2, 3$; $l = 1, 2, 3$).*

To simplify the proof we write

$$a_1 b_2 c_3 = \alpha_0, \qquad a_1 b_3 c_2 = \beta_0,$$

$$a_2 b_3 c_1 = \alpha_1, \qquad a_2 b_1 c_3 = \beta_1,$$

$$a_3 b_1 c_2 = \alpha_2, \qquad a_3 b_2 c_1 = \beta_2.$$

These α_i and β_i have no relation whatsoever to those used earlier in the article. Clearly as long as every letter and every subscript is present in a term we can continue taking out factors of type $a_i b_k c_l$. But some exponents may be zero. In that case suppose i_1 say, is not zero. Then the conditions on the exponents give the inequalities:

$$i_2 < n, \qquad i_3 < n, \qquad j_1 < n, \qquad k_1 < n,$$

$$j_2 + k_2 > 0, \qquad j_3 + k_3 > 0, \qquad j_2 + j_3 > 0, \qquad k_2 + k_3 > 0.$$

Hence, if $j_2 = 0$, $k_2 > 0$ and $j_3 > 0$, and therefore β_0 is a factor.

If $k_3 = 0$, $j_3 > 0$, $k_2 > 0$ and β_0 is a factor.

If $j_2 = 0$ and $k_3 = 0$ at the same time, β_0 is still a factor.

If $j_3 = 0$ or $k_2 = 0$ or both are zero simultaneously, α_0 is a factor. Hence, *if $i_1 \neq 0$, either α_0 or β_0 is a factor.* Taking out the factor α_0 (or β_0), we

have left a term of same type with $n-1$ replacing n. This establishes the lemma and we can write any rational invariant in the form

$$I = \sum \alpha_0^i \alpha_1^j \alpha_2^k \beta_0^l \beta_1^m \beta_2^n.$$

Since I is left unaltered by all permutations of a, b, c and their subscripts, it is left unaltered by all permutations of α and β and their subscripts. For if we express the permutations of the Roman letters in cycles, it is easy to express the corresponding permutations of the Greek letters in cyclic form. We have

$$\begin{aligned}
(a\,b\,c) &\text{ is equivalent to } (\alpha_0\alpha_2\alpha_1)(\beta_0\beta_1\beta_2),\\
(1\,2\,3) &\text{ " " " } (\alpha_0\alpha_1\alpha_2)(\beta_0\beta_1\beta_2),\\
(b\,c) &\text{ " " " } (\alpha_0\beta_0)(\alpha_1\beta_1)(\alpha_2\beta_2),\\
(2\,3) &\text{ " " " } (\alpha_0\beta_0)(\alpha_1\beta_2)(\alpha_2\beta_1).
\end{aligned}$$

This includes all independent types.

Next we introduce the notation:

$$\alpha_0+\alpha_1+\alpha_2 = r_1, \qquad \beta_0+\beta_1+\beta_2 = s_1,$$

$$\alpha_1\alpha_2+\alpha_2\alpha_0+\alpha_0\alpha_1 = r_2, \qquad \beta_1\beta_2+\beta_2\beta_0+\beta_0\beta_1 = s_2,$$

$$\alpha_0\alpha_1\alpha_2 = p_3 = \beta_0\beta_1\beta_2,$$

$$(\alpha_1-\alpha_2)(\alpha_2-\alpha_0)(\alpha_0-\alpha_1) = r, \qquad (\beta_1-\beta_2)(\beta_2-\beta_0)(\beta_0-\beta_1) = s.$$

Since any invariant admits of the permutations $(\alpha_0\alpha_1\alpha_2)$ and $(\beta_0\beta_1\beta_2)$, it follows that if any invariant I contains the term $\alpha_0^i\alpha_1^j\alpha_2^k\beta_0^l\beta_1^m\beta_2^n$, it contains the 9 terms

$$\sum^3 \alpha_0^i\alpha_1^j\alpha_2^k \sum^3 \beta_0^l\beta_1^m\beta_2^n,$$

$\sum^3$ signifying the sum of the three terms obtained by cyclically permuting i, j, k and l, m, n respectively. If $i=j=k$ or $l=m=n$, these nine terms reduce to three, or if both sets of equalities exist simultaneously, they reduce to a single term. In any event, we have the product of two alternating functions of α_0, α_1, α_2 and β_0, β_1, β_2 respectively. By the same argument for all terms of I, we see that

$$I = R(r_1, s_1, r_2, s_2, p_3, r, s), \tag{1}$$

where R is a rational, integral, isobaric function of its arguments. The subscripts indicate the weight, r and s being of weight 3.

As a next step r and s may be assumed to occur to the first degree only, for r^2 and s^2 can both be expressed rationally in terms of the other 5 quantities. In fact*

*SALMON, *Higher Algebra*, Ex. 2, p. 57.

$$r^2 = r_1^2 r_2^2 - 4r_1^3 p_3 + 18 r_1 r_2 p_3 - 4r_2^3 - 27 p_3^2,$$
$$s^2 = s_1^2 s_2^2 - 4s_1^3 p_3 + 18 s_1 s_2 p_3 - 4s_2^3 - 27 p_3^2.$$

Also since the permutation $(\alpha_1\alpha_2)(\beta_1\beta_2)$ changes the sign of r and of s, they can occur only in the combination rs. Hence we can write

$$I = R_1(r_1, s_1, r_2, s_2, p_3) + rs \cdot R_2(r_1, s_1, r_2, s_2, p_3). \tag{2}$$

Again since $(\alpha_0\beta_0)(\alpha_1\beta_1)(\alpha_2\beta_2)$ interchanges r_i and s_i, I must be unaltered by interchanging r_i and s_i. That is

$$R_1(r_1, s_1, r_2, s_2, p_3) = R_1(s_1, r_1, s_2, r_2, p_3). \tag{3}$$

Hence if I contains a term $r_1^i s_1^k r_2^l s_2^m p_3^n$, it contains the two terms

$$p_3^n (r_1 s_1)^k (r_2 s_2)^m (r_1^a r_2^b + s_1^a s_2^b) = A_1, \tag{4}$$

where for convenience we assume $k \leqq i$, $m \leqq l$, $i - k = a$, $l - m = b$.

Now let

$$p_3^n (r_1 s_1)^k (r_2 s_2)^m (r_1^a s_2^b + s_1^a r_2^b) = A_2. \tag{5}$$

Then

$$\begin{aligned} A_1 + A_2 &= p_3^n (r_1 s_1)^k (r_2 s_2)^m (r_1^a + s_1^a)(r_2^b + s_2^b) \\ &= S_1(r_1 + s_1, r_2 + s_2, r_1 s_1, r_2 s_2, p_3), \end{aligned} \tag{6}$$

$$\begin{aligned} A_1 - A_2 &= p_3^n (r_1 s_1)^k (r_2 s_2)^m (r_1^a - s_1^a)(r_2^b - s_2^b) \\ &= (r_1 - s_1)(r_2 - s_2) S_2(r_1 + s_1, r_2 + s_2, r_1 s_1, r_2 s_2, p_3), \end{aligned} \tag{7}$$

where S_1 and S_2 are rational functions. From (6) and (7) we can solve for A_1 in terms of $r_1 - s_1$, $r_2 - s_2$ and the arguments of S_i. Carrying out the same process for all terms of R_1, we get

$$R_1 = M_1(r_1 + s_1, r_2 + s_2, r_1 s_1, r_2 s_2, p_3) + (r_1 - s_1)(r_2 - s_2) M_2(r_1 + s_1, r_2 + s_2, r_1 s_1, r_2 s_2, p_3),$$

where M_i is a rational, integral, isobaric function of its arguments. R_2 can be expressed in the same way. Hence, we have the following theorem.

Any rational simultaneous invariant of T_1 and T_2 is a rational, integral isobaric function of $r_1 + s_1$, $r_2 + s_2$, $r_1 s_1$, $r_2 s_2$, p_3, $(r_1 - s_1)(r_2 - s_2)$, and rs.

Since the above seven arguments are themselves invariants, it follows that they form a rationally complete system. They are not independent, however, for we saw that $\overline{rs}^2$ could be expressed in terms of the other six. It remains to show that these seven can be expressed in terms of the system proposed in the preceding section. The equalities are easily seen to be

$$\begin{aligned} r_1 + s_1 &= I_1, & & \\ r_2 + s_2 &= I_2, & r_2 - s_2 &= D_2, \\ p_3 &= I_3, & r_1 s_1 &= \tfrac{1}{4}(I_1^2 - D_1^2), \\ r_1 - s_1 &= D_1, & r_2 s_2 &= \tfrac{1}{4}(I_2^2 - D_2^2). \end{aligned}$$

The expression for $\overline{rs}^2$ in terms of the present system is very long. Since rs must be added to our system, for *rational completeness* we will call it I_6. In terms of the coefficients of the 3-point

$$I_6 = (a_2b_3c_1 - a_3b_1c_2)(a_3b_1c_2 - a_1b_2c_3)(a_1b_2c_3 - a_2b_3c_1)$$
$$(a_2b_1c_3 - a_3b_2c_1)(a_3b_2c_1 - a_1b_3c_2)(a_1b_3c_2 - a_2b_1c_3).$$

The completeness of the system has thus been finally established. The principle employed in this paragraph, namely, that if an invariant contains a term $\alpha_0^i\alpha_1^j\alpha_2^k\beta_0^l\beta_1^m\beta_2^n$, it contains all terms obtained by permuting the i, j, k and l, m, n, is useful in determining all invariant relations, as we need only concern ourselves about a single term of each type.

§ 3. *The dual forms.*

We next consider the transformation of the invariants when the rôle of the triangles is interchanged, that is, when the one originally taken as a 3-line is regarded as a 3-point and *vice versa*. Obviously in the simplified form of the invariants we need only replace each letter by its minor in D_1. We will designate the transforms by primes. To abbreviate we denote the minor of any letter by the corresponding capital letter.

$$D_1' = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = D_1^2.$$

This follows from the theory of 3-rowed determinants.

$$I_1' = A_1B_2C_3 + A_2B_3C_1 + A_3B_1C_2 + A_1B_3C_2 + A_2B_1C_3 + A_3B_2C_1$$
$$= \sum^3 a_1^2b_2^2c_3^2 - \sum^3 a_1^2b_3^2c_2^2 - 4(\sum^3 a_1a_2b_2b_3c_3c_1 - \sum^3 a_1a_2b_3b_1c_2c_3).$$

This can be expressed easily, for it is of weight 2, and the D's must enter in each term and to an odd degree. This follows from the remark that if the terms of a given type have all the same sign, the D's must enter to an even degree if at all, but if half the terms are positive and half negative, the D's must enter in each term and to an odd degree. Hence in the present case we need consider only I_1, D_1, and D_2. It turns out that

$$I_1' = I_1D_1 - 6D_2.$$

I_2' and D_2' stand in a certain formal relation to a square array,

$$\begin{vmatrix} A_2A_3 & B_2B_3 & C_2C_3 \\ A_3A_1 & B_3B_1 & C_3C_1 \\ A_1A_2 & B_1B_2 & C_1C_2 \end{vmatrix}.$$

While D_2' is the determinant of this array, I_2' is the expanded form with all signs positive. If we designate the sum of the positive terms of D_2' by ϕ and of negative terms by θ,

$$D_2' = \phi - \theta, \qquad I_2' = \phi + \theta.$$

The actual calculation is long and the following table of summations is convenient here and elsewhere:

$$\sum^{3} a_1^4 b_2^4 c_3^4 - \sum^{3} a_1^4 b_3^4 c_2^4 = L' - L'',$$

$$\sum^{9} a_1^4 b_2^3 b_3 c_2 c_3^3 - \sum^{9} a_1^4 b_2 b_3^3 c_2^3 c_3 = M' - M'',$$

$$\sum a_1^3 a_2 b_1 b_2^2 b_3 c_2 c_3^3 - \sum^{9} a_1^3 a_2 b_2 b_3^3 c_1 c_2^2 c_3 = N' - N'',$$

$$\sum^{18} a_1^3 a_2 b_1 b_2 b_3^2 c_2^2 c_3^2 - \sum^{18} a_1^3 a_2 b_2^2 b_3^2 c_1 c_2 c_3^2 = D' - D'',$$

$$\sum^{3} a_1^2 a_2^2 b_2^2 b_3^2 c_3^2 c_1^2 - \sum^{3} a_1^2 a_2^2 b_3^2 b_1^2 c_2^2 c_3^2 = E' - E'',$$

$$\sum^{3} a_1^2 a_2 a_3 b_1 b_2^2 b_3 c_1 c_2 c_3^2 - \sum^{3} a_1^2 a_2 a_3 b_1 b_2 b_3^2 c_1 c_2^2 c_3 = F' - F'',$$

$$\sum^{6} a_1^3 a_2 b_2^3 b_3 c_3^3 c_1 - \sum^{6} a_1^3 a_2 b_3^3 b_1 c_2^3 c_3 = G' - G'',$$

$$\sum^{9} a_1^2 a_2^2 b_1 b_2 b_3^2 c_1 c_2 c_3^2 = H,$$

$$\sum^{9} a_1^4 b_2^2 b_3^2 c_2^2 c_3^2 = K.$$

Further let $L' + L'' = L$; etc. In this notation, there is no difficulty beyond long reckoning. Less direct methods offered no advantage. We have then

$$\phi = E' + 3E'' - 2D' + 2N' + N'' - 5F' + 6F'' - G',$$

$$\theta = 3E' + E'' - 2D'' + N' + 2N'' + 6F'' - 5F''' - G''.$$

Hence

$$I_2' = 4E - 2D + 3N + F - G,$$

$$D_2' = -2(E'-E'') - 2(D'-D'') + (N'-N'') - 11(F'-F''') - (G'-G'').$$

In I_2' the D's must enter to an even degree, if at all. It is readily shown that

$$D_1^2 I_2 = N - 2D + 2E - F + G + 4H,$$

$$I_1 D_1 D_2 = -N + 2E + 5F + G - 4H,$$

$$D_2^2 = E + 2F - 2H,$$

and from these

$$I_2' = D_1^2 I_2 - 2I_1 D_1 D_2 + 6D_2^2.$$

D_2' must contain the D's to odd degree in every term. It turns out at once that

$$D_2' = -D_1^2 D_2,$$
$$I_3' = A_1 A_2 A_3 B_1 B_2 B_3 C_1 C_2 C_3.$$

The actual expansion and reduction of this form is too long for reproduction here, but we get as a final form

$$I_3' = D_1^3 I_3 - D_2^3 + \tfrac{1}{2} D_1 D_2 (I_1 D_2 - I_2 D_1).$$

Summarizing the results of this section into a compact table, we have:

$$I_1' = I_1 D_1 - 6 D_2,$$
$$D_1' = D_1^2,$$
(I) $$I_2' = D_1^2 I_2 - 2 I_1 D_1 D_2 + 6 D_2^2,$$
$$D_2' = -D_1^2 D_2,$$
$$I_3' = D_1^3 I_3 - D_2^3 + \tfrac{1}{2} D_1 D_2 (I_1 D_2 - I_2 D_1).$$

It is a simple matter to write down the corresponding table when the 3-line is taken in general form.*

These formulæ are very useful, for if we know that the vanishing of an invariant I indicates a certain projective relation between the 3-line and the 3-point, the vanishing of I', obtained from I by these formulæ, will indicate the same relation between the joins of the 3-point and the meets of the 3-line.

This dual transformation, being obviously of period 2, affords an easy check on the algebra involved in calculating the dual forms. Carrying out the substitutions of (I) a second time we get

$$I_1'' = D_1^2 (I_1 D_1 - 6 D_2) + 6 D_1^2 D_2 = D_1^3 I_1,$$
$$D_1'' = D_1^4, \qquad D_2'' = D_1^6 D_2,$$
$$I_2'' = D_1^6 I_2, \qquad I_3'' = D_1^9 I_3.$$

This is an excellent verification of the dual formulæ.

§ 4. *Some invariant relations of the three-point and the three-line.*

We shall calculate directly a few of the simpler invariant relations.

(1) *The three points are collinear if*

$$D_1 = 0.$$

*For these forms, see the Johns Hopkins University Circular, July, 1908, p. 125.

(2) *The three lines are concurrent if*

$$\Delta_1 = 0.$$

The above two theorems require no formal proof.

(3) *The 3-point is apolar to the 3-line if*

$$I_1 = 0.$$

For $x_1x_2x_3 = 0$ is the 3-line and $(a\xi)(b\xi)(c\xi) = 0$ the 3-point. Regarding the x's as differential operators, and operating on the equation of the 3-point, we get merely the coefficient of $\xi_1\xi_2\xi_3$ in the equation, which must be zero for apolarity. This is,

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 + a_1b_3c_2 + a_2b_1c_3 + a_3b_2c_1 = 0.$$

But that is precisely

$$I_1 = 0.$$

(4) *The three points of the 3-point and the meets of the 3-line are on a conic if*

$$D_2 = 0.$$

For the general conic on the meets of the 3-line, i. e., on the vertices of the reference triangle, is

$$\alpha_1x_2x_3 + \alpha_2x_3x_1 + \alpha_3x_1x_2 = 0.$$

If the points $(a\xi) = 0$, $(b\xi) = 0$ and $(c\xi) = 0$ are on this conic;

$$\alpha_1a_2a_3 + \alpha_2a_1a_3 + \alpha_3a_1a_2 = 0,$$
$$\alpha_1b_2b_3 + \alpha_2b_1b_3 + \alpha_3b_1b_2 = 0,$$
$$\alpha_1c_2c_3 + \alpha_2c_1c_3 + \alpha_3c_1c_2 = 0.$$

For these 3 equations to be consistent, their determinant must vanish. That is

$$\begin{vmatrix} a_2a_3 & a_3a_1 & a_1a_2 \\ b_2b_3 & b_3b_1 & b_1b_2 \\ c_2c_3 & c_3c_1 & c_1c_2 \end{vmatrix} = 0,$$

which is $D_2 = 0$. For the general case, a study of the degree in the Greek letters shows that

$$\Delta_1^2 D_2 = 0$$

is the general condition.

The dual of D_2, $D_1^2 D_2 = 0$, is therefore the condition that the 6 lines be on a conic.

(5) We will next take the problem of PASCH* triangles and prove: *If the*

*Mathematische Annalen, Vol. 23 (1884), p. 426 ff. See also HUN, these Transactions, Vol. 5 (1904), p. 49. PASCH shows that if there is one such triangle in a collineation, there is a quadruple infinity.

two triangles are such that the 3*-point is a* PASCH *triangle in a normal collineation having the* 3*-line for a fixed triangle,*

$$D_1(I_1 D_1 - 3D_2) = 0.$$

To show this, we take again the 3-line as reference triangle. The normal collineation can be written

$$x_i = ka_i$$

and we want the transform of a to be on the line bc. Similarly b is to go into a point on ac, and c into a point on ab. This gives us three equations like

$$\begin{vmatrix} k_1 a_1 & k_2 a_2 & k_3 a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

We can write them

$$k_1 a_1 A_1 + k_2 a_2 A_2 + k_3 a_3 A_3 = 0,$$
$$k_1 b_1 B_1 + k_2 b_2 B_2 + k_3 b_3 B_3 = 0,$$
$$k_1 c_1 C_1 + k_2 c_2 C_2 + k_3 c_3 C_3 = 0,$$

where the A_i, B_i, C_i, are minors as before.

For these 3 equations to be consistent, we must have

$$\begin{vmatrix} a_1 A_1 & a_2 A_2 & a_3 A_3 \\ b_1 B_1 & b_2 B_2 & b_3 B_3 \\ c_1 C_1 & c_2 C_2 & c_3 C_3 \end{vmatrix} = 0.$$

Expanding this we get

$$\sum^{6} a_1^3 b_2^2 c_3^3 - \sum^{18} a_1^3 b_2^2 b_3 c_2 c_3^2 + \sum^{18} a_1^2 a_3 b_2^2 b_3 c_1 c_2 c_3 - 6 a_1 a_2 a_3 b_1 b_2 b_3 c_1 c_2 c_3 = 0.$$

This expresses itself readily in the form

$$ID_1^2 - 3D_1 D_2 = 0, \quad \text{or} \quad D_1(I_1 D_1 - 3D_2) = 0,$$

as we were to prove.

(6) *If the two triangles are in perspective position,*

$$I_6 = 0.$$

It will be found that the 6 factors of I_6 correspond to the 6 possible orderings of the vertices of the two triangles. For example, by way of proof, suppose we ask that the joins of 1, 0, 0; 0, 1, 0; 0, 0, 1 to a; b; c, respectively meet in a point. The 3 joins are

$$a_2 x_3 - a_3 x_2 = 0,$$
$$b_3 x_1 - b_1 x_3 = 0,$$
$$c_1 x_2 - c_2 x_1 = 0.$$

Hence the condition that they meet in a point is

$$\begin{vmatrix} 0 & -a_3 & a_2 \\ b_3 & 0 & -b_1 \\ -c_2 & c_1 & 0 \end{vmatrix} = 0,$$

or $a_2b_3c_1 - a_3b_1c_2 = 0$. In the same way each ordering requires a distinct factor of I_6 to vanish. As a corollary we can also state: *If there exists a conic with respect to which the 3-line and 3-point are mutually polar, then must*

$$I_6 = 0.$$

All the problems discussed by Dr. HUN might be treated independently as those of this paragraph have been. But every fundamental system of invariants must be expressible in terms of the present system, since it has been proven complete. Hence by constructing a table of equivalents we can utilize directly Dr. HUN's results.

§ 5. *Hun's invariants in terms of the present system.*

HUN's three fundamental invariants are defined for the case of a general 3-line and 3-point as follows:

$$I_1 = -\sum B_{11},$$
$$I_2 = \sum (B_{22}B_{33} - B_{23}B_{32}),$$
$$I_3 = -\begin{vmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{vmatrix};$$

where

$$B_{ii} = \alpha_i(\beta\gamma)_i + \beta_i(\gamma\alpha)_i + \gamma_i(\alpha\beta)_i,$$
$$B_{ik} = \alpha_i(\beta\gamma)_k + \beta_i(\gamma\alpha)_k + \gamma_i(\alpha\beta)_k,$$

and in turn the symbols $(\beta\gamma)_i$, $(\gamma\alpha)_k$, etc., are defined as follows:

$$(\beta\gamma)_i = a_i(bc/\beta\gamma) + b_i(ca/\beta\gamma) + c_i(ab/\beta\gamma);$$

where finally

$$(bc/\beta\gamma) = (b\beta)(c\gamma) + (b\gamma)(c\beta),$$
$$(ca/\beta\gamma) = (c\beta)(a\gamma) + (c\gamma)(a\beta), \text{ etc.},$$

the parentheses now indicating row products.

If now the 3-line is taken as the reference triangle, we need retain only terms containing the product $\alpha_1\beta_2\gamma_3$ and replace it by unity. This gives

$$B_{11} = B_{22} = B_{33} = \sum^{6} a_i b_k c_l,$$

$$B_{21} = 2(a_1b_1c_3 + a_1b_3c_1 + a_3b_1c_1),$$

$$B_{12} = 2(a_2b_2c_3 + a_2b_3c_2 + a_3b_2c_2).$$

In general

$$B_{ij} = 2(a_jb_jc_k + a_jb_kc_j + a_kb_jc_j). \tag{1}$$

Then denoting HUN's invariants by bars to avoid confusion, we have

$$\begin{aligned}
\bar{I}_1 &= -\sum^{3} B_{ii} = -3\sum^{6} a_ib_kc_l = -3I_1,\\
\bar{I}_2 &= \sum^{3}(B_{22}B_{33} - B_{23}B_{32}) = 3(\sum^{6} a_ib_kc_l)^2\\
&\quad - 4[(a_3b_3c_1 + a_3b_1c_3 + a_1b_3c_3)(a_2b_2c_1 + a_2b_1c_2 + a_1b_2c_2)\\
&\quad + (a_3b_3c_2 + a_3b_2c_3 + a_2b_3c_3)(a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1)\\
&\quad + (a_2b_2c_3 + a_2b_3c_2 + a_3b_2c_2)(a_1b_1c_3 + a_1b_3c_1 + a_3b_1c_1)]\\
&= 3I_1^2 - 4(\sum^{9} a_1^2b_2b_3c_2c_3 + 3\sum^{6} a_2a_3b_3b_1c_1c_2)\\
&= 3I_1^2 - (I_1^2 - D_1^2 + 12I_2) = 2I_1^2 + D_1^2 - 12I_2.
\end{aligned}$$

The transformation of $-I_3 = -\bar{\Delta}\bar{D}\bar{N}$ is more difficult. Obviously $\bar{\Delta}$ and $\bar{D}$ are Δ_1 and D_1 respectively. We have directly from above equalities (1),

$$-I_3 = \begin{vmatrix} I_1 & 2(a_2b_2c_3+a_2b_3c_2+a_3b_2c_2) & 2(a_3b_3c_2+a_3b_2c_3+a_2b_3c_3)\\ 2(a_1b_1c_3+a_1b_3c_1+a_3b_1c_1) & I_1 & 2(a_3b_3c_1+a_3b_1c_3+a_1b_3c_3)\\ 2(a_1b_1c_2+a_1b_2c_1+a_2b_1c_1) & 2(a_2b_2c_1+a_2b_1c_2+a_1b_2c_2) & I_1 \end{vmatrix}.$$

Expanding this and using the equality found in evaluating $\bar{I}_2$, we have

$$\begin{aligned}
-\bar{I}_3 &= I_1^3 + 16\sum^{18} a_1a_2a_3b_2b_3^2c_1^2c_2 + 8\sum^{12} a_2a_3^2b_3b_1^2c_1c_2^2\\
&\quad + 48a_1a_2a_3b_1b_2b_3c_1c_2c_3 - I_1(I_1^2 - D_1^2 + 12I_2)\\
&= I_1^3 + 12I_1I_2 - 4D_1D_2 - I_1^3 + I_1D_1^2 - 12I_1I_2 = D_1(I_1D_1 - 4D_2).
\end{aligned}$$

Hence

$$\bar{N} = 4D_2 - I_1D_1.$$

Therefore we have as a complete table

$$\begin{aligned}
\bar{D} &= D_1, \qquad & \bar{I}_2 &= 2I_1^2 + D_1^2\Delta_1^2 - 12I_2,\\
\bar{\Delta} &= \Delta_1, \qquad & \bar{N} &= -I_1D_1\Delta_1 + 4D_2,\\
& & \bar{I}_1 &= -3I_1,
\end{aligned}$$

where the Δ_1's are inserted so as to be correct for the general 3-line.

Using these formulas we shall write down the equivalents of Dr. HUN's results as tabulated on pages 50 and 51 of these Transactions, vol. 5 (1904).

The 3-point degenerates, if

$$D_1 = 0.$$

The 3-point is apolar to the 3-line, if

$$I_1 = 0.$$

The meets of the 3-line are apolar to the joins of the 3-point, if

$$I_1 D_1 \Delta_1 - 6D_2 = 0.$$

A conic circumscribed to the 3-point, and apolar to the 3-line exists, if

$$I_1 \Delta_1 D_1 - 2D_2 = 0.$$

A point conic apolar to both the 3-point and the 3-line exists, if

$$D_1^2 D_2 = 0.$$

A conic circumscribed about both the 3-point and the 3-line exists, if

$$\Delta_1^2 D_2 = 0.$$

There exists a point, whose polar conic as to the 3-line is apolar to the 3-point, if

$$D_1 \Delta_1 (I_1 D_1 \Delta_1 - 4D_2) = 0.$$

The three polar lines as to the 3-line, of the points of the 3-point, taken two at a time, meet in a point, if

$$\Delta_1 (I_1 D_1 \Delta_1 - 4D_2) = 0.$$

There exists a collineation having the 3-point as a fixed triangle, which sends each meet of the 3-line into a point on the opposite line, if

$$\Delta_1 (I_1 D_1 \Delta_1 - 3D_2) = 0.$$

There exists a collineation having the meets of the 3-line as fixed points, which sends each point of the 3-point into a

The 3-line degenerates, if

$$\Delta_1 = 0.$$

The 3-line is apolar to the 3-point, if

$$I_1 = 0.$$

The joins of the 3-point are apolar to the meets of the 3-line, if

$$I_1 D_1 \Delta_1 - 6D_2 = 0.$$

A conic inscribed in the 3-line and apolar to the 3-point exists, if

$$I_1 D_1 \Delta_1 - 2D_2 = 0.$$

A line conic apolar to both the 3-line and the 3-point exists, if

$$\Delta_1^2 D_2 = 0.$$

A conic inscribed to both the 3-point and the 3-line exists, if

$$D_1^2 D_2 = 0.$$

There exists a line, whose polar conic as to the 3-point is apolar to the 3-line, if

$$D_1 \Delta_1 (I_1 D_1 \Delta_1 - 4D_2) = 0.$$

The three polar points as to the 3-point, of the lines of the 3-line, taken two at a time, lie on a line, if

$$D_1 (I_1 D_1 \Delta_1 - 4D_2) = 0.$$

There exists a collineation having the 3-line as a fixed triangle, which sends each join of the 3-point into a line through the opposite point, if

$$D_1 (I_1 D_1 \Delta_1 - 3D_2) = 0.$$

There exists a collineation having the joins of the 3-point as fixed lines, which sends each line of the 3-line into a line

point on the join of the other two, if

$$I_1 D_1 \Delta_1 - 3D_2 = 0.$$

There exists a point, such that its polar conic as to the joins of the 3-point is apolar to the meets of the 3-line, if

$$D_1^2 \Delta_1^2 (I_1 D_1 \Delta_1 - 2D_2) = 0.$$

through the meet of the other two, if

$$I_1 D_1 \Delta_1 - 3D_2 = 0.$$

There exists a line such that its polar conic as to the meets of the 3-line is apolar to the joins of the 3-point, if

$$D_1^2 \Delta_1^2 (I_1 D_1 \Delta_1 - 2D_2) = 0.$$

To this we will add one more, adapted from the same article.

There exists a line conic, touching the joins of the 3*-point, and apolar to the* 3*-line, if*

$$D_1^2 (I_1 D_1 \Delta_1 - 4D_2) = 0.$$

It will be seen that the essential factor of all these invariants is a member of the pencil,

$$I_1 D_1 \Delta_1 + \lambda D_2 = 0.$$

It may be noted that if two points of the 3-point are taken as the circular points at infinity, and the third point taken as variable, this pencil is the pencil of circles determined by the Feuerbach circle and the circumcircle.

Another pair of dual theorems which can be readily proven are these:

If one of the 3 *points lies on either of the lines of the* 3*-line,*

$$I_3 = 0.$$

If a meet of the 3*-line lies on a join of the* 3*-point,*

$$2D_1^3 - 2D_2^3 + D_1 D_2 (I_1 D_2 - I_2 D_1) = 0.$$

The powers of Δ_1 are not inserted in the latter form, and will not be in the future unless for some special reason.

§ 6. *A conic inscribed in one triangle and circumscribed about the other.*

The present investigation has grown by extension from the problem: To find the invariant relation on two triangles so that a conic may be drawn touching the sides of one triangle and on the vertices of the other. This problem together with its applications is especially interesting.

Let the three lines be the reference triangle, and take the polar conics of the three points a, b, c as to it. The three polar conics are:

$$a_1 x_2 x_3 + a_2 x_3 x_1 + a_3 x_1 x_2 = 0, \tag{1}$$

$$b_1 x_2 x_3 + b_2 x_3 x_1 + b_3 x_1 x_2 = 0, \tag{2}$$

$$c_1 x_2 x_3 + c_2 x_3 x_1 + c_3 x_1 x_2 = 0. \tag{3}$$

Let η be any line of (1), the polar conic of a, and the *poloconic* of η will pass through a.* Hence if the three polar conics have η as a common line, then the poloconic of η as to the 3-line will be on all three points a, b, c. But it will also touch the three reference lines α, β, γ, for the Hessian of a triangle is the triangle itself, and the poloconic of any cubic touches the Hessian in three points†; in the present case it touches each of the three lines making up the Hessian, for they enter symmetrically.

Consequently the problem is now reduced to finding the condition that the three polar conics have a common line. SALMON‡ gives the condition that three point conics have a common point as $T^2 = 64M$. For three conics to have a common line we need only form their line equations and impose the analogous condition. The line equations of (1), (2), (3) are

$$(4)\quad a_1^2\xi_1^2 + a_2^2\xi_2^2 + a_3^2\xi_3^2 - 2a_2a_3\xi_2\xi_3 - 2a_3a_1\xi_3\xi_1 - 2a_1a_2\xi_1\xi_2 = 0 = U,$$

$$(5)\quad b_1^2\xi_1^2 + b_2^2\xi_2^2 + b_3^2\xi_3^2 - 2b_2b_3\xi_2\xi_3 - 2b_3b_1\xi_3\xi_1 - 2b_1b_2\xi_1\xi_2 = 0 = V,$$

$$(6)\quad c_1^2\xi_1^2 + c_2^2\xi_2^2 + c_3^2\xi_3^2 - 2c_2c_3\xi_2\xi_3 - 2c_3c_1\xi_3\xi_1 - 2c_1c_2\xi_1\xi_2 = 0 = W.$$

Now $M = 0$ is the condition that in a net of line conics $lU + mV + nW$ there shall be a double point. In that case, the reciprocal of that particular conic of the net vanishes identically. Following SALMON's notation,§ we find the equation to be

$$(7)\qquad l^2\Sigma + m^2\Sigma' + n^2\Sigma'' + mn\phi_{23} + nl\phi_{31} + lm\phi_{12} = 0.$$

The Σ's are the reciprocals of U, V, W respectively and the ϕ_{ik} are the Clebschians of the same taken two at a time:

$$(8)\qquad \Sigma = 4a_1a_2a_3(a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2) = 0.$$

Σ' and Σ'' are the same in b and c respectively. By the Clebschian of two line conics $(a\xi)^2$ and $(b\xi)^2$, we mean the contravariant $|abx|^2$, where $|abx|$ is the determinant.‖ In the present case

$$\begin{aligned}(9)\qquad \phi_{23} &= (a_2b_3 - a_3b_2)^2x_1^2 + (a_3b_1 - a_1b_3)^2x_2^2 + (a_1b_2 - a_2b_1)^2x_3^2\\ &+ (a_3b_1 - a_1b_3)(a_1b_2 - a_2b_1)x_2x_3 + (a_1b_2 - a_2b_1)(a_2b_3 - a_3b_2)x_1x_3\\ &+ (a_2b_3 - a_3b_2)(a_3b_1 - a_1b_3)x_1x_2,\end{aligned}$$

*SALMON-FIEDLER, — *Analytische Geometrie der höheren ebenen Kurven*, p. 292. The English edition of this book is out of print. SALMON calls the poloconic simply the polar conic of the line. CLEBSCH-LINDEMANN, — *Vorlesungen über Geometrie*, Vol. I, p. 543. KOHN, *Encyklopädie der mathematischen Wissenschaften*, Vol. III, part 4, p. 471, gives an account of the poloconic with references.

† DURÈGE: *Ebene Curven dritter Ordnung*, p. 278.

‡ *Conic Sections*, p. 365–6.

§ *Conic Sections*, p. 366.

‖ Ibid., p. 344. CLEBSCH's form is non-symbolic and ϕ equals ϕ_{12} above. See also CLEBSCH-LINDEMANN, Vol. I, p. 277. If $a_i \equiv b_i$, the Clebschian is the line equation.

and ϕ_{31} and ϕ_{12} are similar forms in b, c and c, a respectively. The above forms, $\sum$ and ϕ_{ik}, are most easily verified by expanding the symbolic forms and substituting the real coefficients.

SALMON gives M in the form of a 6-rowed determinant. From equations (8) we see that the block

$$\begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix}$$

is composed entirely of zeros. Hence M reduces at once to the product of two 3-rowed determinants

$$(10)\quad \begin{vmatrix} 2a_1^2a_2a_3 & 2a_1a_2^2a_3 & 2a_1a_2a_3^2 \\ 2b_1^2b_2b_3 & 2b_1b_2^2b_3 & 2b_1b_2b_3^2 \\ 2c_1^2c_2c_3 & 2c_1c_2^2c_3 & 2c_1c_2c_3^2 \end{vmatrix} \times \begin{vmatrix} (a_2b_3-a_3b_2)^2 & (a_3b_1-a_1b_3)^2 & (a_1b_2-a_2b_1)^2 \\ (b_2c_3-b_3c_2)^2 & (b_3c_1-b_1c_3)^2 & (b_1c_2-b_2c_1)^2 \\ (c_2a_3-c_3a_2)^2 & (c_3a_1-c_1a_3)^2 & (c_1a_2-c_2a_1)^2 \end{vmatrix}.$$

The first determinant is clearly $8I_3D_1$. The second is seen to be the dual of

$$(11)\quad \begin{vmatrix} c_1^2 & c_2^2 & c_3^2 \\ a_1^2 & a_2^2 & a_3^2 \\ b_1^2 & b_2^2 & b_3^2 \end{vmatrix}.$$

Now (11) in terms of our fundamental invariants is

$$I_1D_1 - 2D_2.$$

The dual of this, by the formulæ of (I), § 3, is

$$D_1^2(I_1D_1 - 4D_2).$$

Therefore

$$(12)\quad M = 8I_3D_1^3(I_1D_1 - 4D_2).$$

To express T similarly we can start with either of SALMON's forms as given on pages 365 and 367 of his *Conic Sections*. We shall develop the second form

$$(13)\quad T = \theta_{123}^2 - 4(\theta_{122}\theta_{133} + \theta_{211}\theta_{233} + \theta_{311}\theta_{322}) + 12\Theta.$$

Next recall the meaning of the θ's. If we write the discriminant of

$$(14)\quad l(a\xi)^2 + m(b\xi)^2 + n(c\xi)^2$$

we can define them as follows: θ_{123} is the coefficient of lmn in the discriminant, θ_{122} is the coefficient of lm^2, and so on. For the net $lU + mV + nW$, this gives

$$(15)\quad \theta_{123} = \sum^6 a_1^2b_2^2c_3^2 - 2\sum^6 a_1a_2b_2b_3c_3c_1 - 2\sum^9 a_1^2b_2b_3c_2c_3 = D_1^2 - 4I_2,$$

$$\theta_{122} = -4(a_1a_2b_1b_2b_3^2 + a_2a_3b_2b_3b_1^2 + a_3a_1b_1b_2^2b_3), \tag{16}$$

$$\theta_{133} = -4(a_1a_2c_1c_2c_3^2 + a_2a_3c_1^2c_2c_3 + a_3a_1c_1c_2^2c_3), \tag{17}$$

and similarly for θ_{211}, θ_{233}, θ_{311}, θ_{322}. Combining these we have

$$\theta_{122}\theta_{133} = 16(a_1^2a_2^2b_1b_2b_3^2c_1c_2c_3^2 + a_2^2a_3^2b_1^2b_2b_3c_1^2c_2c_3 + a_3^2a_1^2b_1b_2^2b_3c_1c_2^2c_3 + \sum^{6} a_1^2a_2a_3b_1b_2^2b_3c_1c_2c_3^2),$$

$$\theta_{211}\theta_{233} = 16(a_1a_2a_3^2b_1^2b_2^2c_1c_2c_3^2 + a_1^2a_2a_3b_2^2b_3^2c_1^2c_2c_3 + a_1a_2^2a_3b_3^2b_1^2c_1c_2^2c_3 + \sum^{6} a_1^2a_2a_3b_1b_2^2b_3c_1c_2c_3^2),$$

$$\theta_{311}\theta_{322} = 16(a_1a_2a_3^2b_1b_2b_3^2c_1^2c_2^2 + a_1^2a_2a_3b_1^2b_2b_3c_2^2c_3^2 + a_1a_2^2a_3b_1b_2^2b_3c_3^2c_1^2 + \sum^{6} a_1^2a_2a_3b_1b_2^2b_3c_1c_2c_3^2).$$

Hence

$$\theta_{122}\theta_{133} + \theta_{211}\theta_{233} + \theta_{311}\theta_{322} = 16(\sum^{9} a_1^2a_2^2b_1b_2b_3^2c_1c_2c_3^2 + 3\sum^{6} a_1^2a_2a_3b_1b_2^2b_3c_1c_2c_3^2) = 4I_2^2 - 4D_2^2 + 48I_1I_3. \tag{18}$$

As a final step we must evaluate Θ, which is defined as the coefficient of lmn in the reciprocal system

$$l\Sigma + m\Sigma' + n\Sigma'' = 0.$$

The expressions for Σ, Σ', Σ'' were given by (8). The discriminant can be written at once in determinant form, as

$$8a_1a_2a_3b_1b_2b_3c_1c_2c_3 \begin{vmatrix} 0 & la_3 + mb_3 + nc_3 & la_2 + mb_2 + nc_2 \\ la_3 + mb_3 + nc_3 & 0 & la_1 + mb_1 + nc_1 \\ a_2 + mb_2 + nc_2 & la_1 + mb_1 + nc_1 & 0 \end{vmatrix} \tag{19}$$

$$= 16a_1a_2a_3b_1b_2b_3c_1c_2c_3(la_1 + mb_1 + nc_1)(la_2 + mb_2 + nc_2)(la_3 + mb_3 + nc_3). \tag{20}$$

The coefficient of lmn in (20) is seen to be

$$16a_1a_2a_3b_1b_2b_3c_1c_2c_3(\sum^{6} a_1b_2c_3) = 16I_1I_3.$$

Hence

$$\Theta = 16I_1I_3. \tag{21}$$

Substituting the values (15), (18) and (21) in (13), we have after reduction

$$T = D_1^4 - 8D_1^2I_2 + 16D_2^2. \tag{22}$$

Substituting the values of M and T thus found, in the condition $T^2 = 64M$, we have the theorem

If the 3*-point and the* 3*-line are such that a conic may be inscribed in the* 3*-line and circumscribed about the* 3*-point, then will*

$$(23) \qquad (D_1^4 - 8D_1^2 I_2 + 16D_2^2)^2 - 512 I_3 D_1^3 (I_1 D_1 - 4D_2) = 0.*$$

Forming the dual of (23) we get a theorem which admits of an important application. The dual is

$$[D_1^8 - 8D_1^4(D_1^2 I_2 - 2I_1 D_1 D_2 + 6D_2^2) + 16 D_1^4 D_2^2]^2$$
$$- 512 D_1^6 (D_1^3 I_3 - D_2^3 + \tfrac{1}{2} I_1 D_1 D_2^2 - \tfrac{1}{2} I_2 D_1^2 D_2)(I_1 D_1^3 - 2D_1^2 D_2) = 0,$$

which reduces to

$$(24) \qquad D_1^{11}[D_1(D_1^4 + 64 I_2^2 + 32 I_1 D_1 D_2 - 16 D_1^2 I_2 - 64 D_2^2)$$
$$- 512 I_3 (I_1 D_1 - 2D_2)] = 0.$$

From this we have the dual theorem, neglecting the trivial case where the three points are on a line:

A conic may be drawn on the meets of the 3*-line and touching the joins of the* 3*-point, if*

$$(26) \quad D_1(D_1^4 + 64 I_2^2 + 32 I_1 D_1 D_2 - 16 D_1^2 I_2 - 64 D_2^2) - 512 I_3 (I_1 D_1 - 2D_2) = 0.$$

Next consider equations (24) and (26) when two points of the 3-point are taken at the circular points at infinity, and the third point as a variable point x. They become equations of curves of the 8th and 5th degrees respectively. We shall discuss these curves briefly, as many properties can be deduced directly. We will designate them by R and Q respectively. The equations of these curves can be written down explicitly by replacing b_i and c_i by the coördinates of I and J and a_i by x_i.

In the case of conics inscribed in the 3-line and on the 3-point, if two of the points are taken at I and J, all the conics become circles touching the 3-line. In other words; *the octavic,* $R = 0$, *is the locus of points from which circles can be drawn touching the lines of the* 3*-line.*

In other words, $R = 0$ is simply the equation of the 4 circles which can be drawn touching the 3 lines.

In the dual case, the join of I and J, that is, the line at infinity, becomes a tangent to all the conics. Hence they are parabolas. Further the lines joining I and J to x are to be tangents to the conics, so that x is in every case a focus. Hence the theorem:

The locus of the foci of parabolas on three points is a quintic curve, and if the points are taken as vertices of the reference triangle, the equation of this quintic is

$$Q = 0.$$

It is possible to deduce many properties of R and Q without writing out the

* This was further verified by expanding SALMON's alternative form, *Conic Sections*, p. 365.

explicit equations. As a first step it is well to note what curves are represented by equating certain simple invariants to zero, when the points are taken as above.

$I_3 = 0$, is the equation of the reference triangle.

$D_1 = 0$, is the line at infinity.

$I_1 = 0$, is the polar line of I and J as to the 3-line.

$D_2 = 0$, is the circumcircle.

$I_1 D_1 - 2D_2 = 0$, is the apolar circle.

$I_1 D_1 - 4D_2 = 0$, is the Feuerbach circle.

None of these require proof here except possibly the last. This can be shown directly from HUN's form, for he shows * that $N = 0$ is the equation of the Feuerbach circle, and we have found $N = I_1 D_1 - 4D_2$. Or it can be proved directly from the easily established fact that $I_1 D_1 - 4D_2 = 0$ is the locus of centers of rectangular hyperbolas on the vertices of the 3-line. But the feet of the three perpendiculars of the 3-line are clearly such centers and are also on the Feuerbach circle. Hence since $I_1 D_1 - 4D_2 = 0$ is a circle, it must be the Feuerbach circle. (All the conics $I_1 D_1 + \lambda D_2 = 0$ are circles, being on I and J, the intersections of $D_1 = 0$ and $D_2 = 0$).

Now take the equation

$$(27) \qquad R = (D_1^4 - 8D_1^2 I_2 + 16D_2^2)^2 - 512 I_3 D_1^3 (I_1 D_1 - 4D_2) = 0,$$

which we know to be the equation of the four touching circles.

If $I_3 = 0$, $(D_1^4 - 8D_1^2 I_2 + 16D_2^2)^2 = 0$. Hence, *the reference lines touch the octavic where they cut K, the quartic*, $D_1^4 - 8D_1^2 I_2 + 16D_2^2 = 0$.

If $I_1 D_1 - 4D_2 = 0$, $K^2 = 0$. Hence, *the Feuerbach circle touches the octavic (or passes through double points) where it cuts K.* This is really the well known theorem that the Feuerbach circle touches the inscribed and escribed circles.

The quartic K is interesting in itself. From its equation,

$$K = D_1^4 - 8D_1^2 I_2 + 16D_2^2 = 0,$$

we see that if $D_1 = 0$, $D_2^2 = 0$, and if $D_2 = 0$, either $D_1^2 = 0$ or $D_1^2 - 8I_2 = 0$. This shows that K is a bicircular quartic, that is, it has double points at I and J. Since the latter are quadruple points of R, all the intersections of K and R are accounted for and easily constructed. There are 12 at the points where the reference triangle touches the R circles, 4 where the Feuerbach circle touches them, and 8 at each of the circular points at infinity.

Since $R = 0$ can be easily constructed, we will now take up $Q = 0$. We can abbreviate

$$Q = D_1[(D_1^2 - 8I_2)^2 + 32 I_1 D_1 D_2 - 64 D_2^2] - 512 I_3 (I_1 D_1 - 2D_2) = 0$$

* These Transactions, vol. 5 (1904), p. 47.

by calling

$$D_1^2 - 8I_2 = U, \qquad 16(I_1 D_1 - 2D_2) = V,$$

therefore

$$Q = D_1(U^2 - D_2 V) - 32 I_3 V = 0. \tag{28}$$

In this if $D_1 = 0$, $I_3 V = 0$. That is, Q cuts the line at infinity where $I_3 V = 0$ cuts it. But $I_3 = 0$ is the reference 3-line and $V = 0$ the apolar circle. Hence

The quintic $Q = 0$, passes through the circular points at infinity and has its asymptotes parallel to the sides of the reference 3-line.

Again if $V = 0$, $D_1 U^2 = 0$. This says that the apolar circle touches the quintic (or cuts it at double points) where U and V intersect. Since Q can be proven rational,* its six double points will not in general lie on a conic. Hence V does not cut Q in double points and we have the theorem:

The apolar circle, $V = 0$, has fourfold contact with the quintic, $Q = 0$, touching it at the four points where $U = 0$ cuts $V = 0$.

This theorem can be proven more directly by taking the dual of $V = 0$, which is $I_1 D_1 - 4D_2 = 0$, or the Feuerbach circle, F. But, by the well-known theorem, the Feuerbach circle touches each of the four tangent circles, R. Since contact is not destroyed in taking duals, V, the dual of F touches Q, the dual of R in four points which was the theorem to be verified. Rewrite Q, thus:

$$Q = D_1 U^2 + 2V(D_1 D_2 - 16 I_3) = D_1 U^2 - 2VW = 0. \tag{29}$$

The curve $W = 0$ is a cubic on the vertices of the reference triangle, cutting the line at infinity where the sides of the reference triangle meet it. That is, its asymptotes are parallel to sides of reference triangle.

When $W = 0$, $U^2 = 0$ or $D_1 = 0$, hence $W = 0$ *touches Q where U cuts W.* But since W and Q have parallel asymptotes, it follows that three of the contacts are at infinity.

We can also find where Q cuts the circumcircle. In Q, let $D_2 = 0$, and it becomes

$$D_1(D_1^2 - 8I_2)^2 - 512 I_1 I_3 D_1 = 0. \tag{30}$$

So unless the 3-point degenerates

$$(D_1^2 - 8I_2)^2 - 512 I_1 I_3 = 0. \tag{31}$$

Then let $D_2 = 0$ in R and drop D_1^4 as a factor, the result,

$$(D_1^2 - 8I_2)^2 - 512 I_1 I_3 = 0,$$

*R. F. Davis, Educational Times, December 2, 1907, p. 545.

is the same as (31). Hence:

Q and R cut the circumcircle in the same points.

It should be noted in this connection that the question, whether a locus is on any of the 6 fundamental points — that is, the points of the 3-point and the meets of the 3-line — is equivalent to the question whether the corresponding invariant vanishes, when there is a coincidence among the fundamental points. This is a question of some importance, but will not be discussed in detail here.

§ 7. *The Clebschians and their invariants.*

In the case of two general line cubics $(a\xi)^3 = 0$ and $(b\xi)^3 = 0$, the Clebschian is a point cubic defined by

$$\chi = |abx|^3 = 0. \tag{1}$$

Similarly the Clebschian of two point cubics $(\alpha x)^3 = 0$ and $(\beta x)^3 = 0$ is defined by

$$\chi' = |\alpha\beta\xi|^3 = 0. \tag{2}$$

If, as in the case of two triangles, the curves are of both the third class and third order, the two Clebschians are in general different. In forming the Clebschians both triangles or cubics must be taken in points or both in lines. So the work here is slightly different from the preceding sections.

If one cubic is taken as the reference 3-point and the other as $(a\xi)^3 = 0$, from (1) we have

$$\chi = 6(a_2a_3^2x_1^2x_2 + a_1a_2^2x_1x_3^2 + a_1^2a_3x_2^2x_3 - a_1a_3^2x_1x_2^2 - a_1^2a_2x_2x_3^2 - a_2^2a_3x_1^2x_3). \tag{3}$$

If in turn the second cubic is taken as the 3-point $(a\xi)(b\xi)(c\xi) = 0$ we must replace

$$3a_ka_l^2 \text{ by } (a_kb_lc_l + a_lb_kc_l + a_lb_lc_k),$$

and this gives us

$$\begin{aligned}\chi = 2[&(a_2b_3c_3 + a_3b_2c_3 + a_3b_3c_2)x_1^2x_2 + (a_1b_2c_2 + a_2b_1c_2 + a_2b_2c_1)x_1x_3^2\\ &+ (a_3b_1c_1 + a_1b_3c_1 + a_1b_1c_3)x_2^2x_3 - (a_1b_3c_3 + a_3b_1c_3 + a_3b_3c_1)x_1x_2^2\\ &- (a_2b_1c_1 + a_1b_2c_1 + a_1b_1c_2)x_2x_3^2 - (a_3b_2c_2 + a_2b_3c_2 + a_2b_2c_3)x_1^2x_3].\end{aligned} \tag{4}$$

It will aid in clearness to define χ geometrically. If we call the reference 3-point d, e, f, then χ may be defined as the locus of points y such that the two triads of lines $y - d, e, f$ and $y - a, b, c$ are apolar. χ' is the locus of lines η having the dual property. From (4), we see that χ is on the vertices of the reference 3-point, and, since the two 3-points enter symmetrically, we have the theorem:

The Clebschian of two 3*-points is on all six points.**

Further since (4) contains no terms in $x_1x_2x_3$, the reference triangle is apolar to it. Hence,

The Clebschian of two 3*-points is a cubic to which both* 3*-points are apolar.*†

The dual theorems for two 3-lines can be written down at once.

Since the invariants, S and T, of χ are mutual invariants of the two 3-points, they must be expressible in terms of our fundamental invariants.

For S, we will make use of SALMON's standard form.† In χ, the SALMON coefficients have the following equivalents,

$$a = b = c = m = 0, \qquad a_2 = (a_2b_3c_3 + a_3b_2c_3 + a_3b_3c_2),$$
$$a_3 = -(a_3b_2c_2 + a_2b_3c_2 + a_2b_2c_3), \qquad b_1 = -(a_1b_3c_3 + a_3b_1c_3 + a_3b_3c_1),$$
$$b_3 = (a_3b_1c_1 + a_1b_3c_1 + a_1b_1c_3), \qquad c_1 = -(a_2b_1c_1 + a_1b_2c_1 + a_1b_1c_2),$$
$$c_2 = (a_1b_2c_2 + a_2b_1c_2 + a_2b_2c_1).$$

The nonessential, common numerical factor has been dropped. This gives, by substituting in SALMON's form,

$$\begin{aligned} S = &-(a_1b_3c_3 + a_3b_1c_3 + a_3b_3c_1)^2(a_1b_2c_2 + a_2b_1c_2 + a_2b_2c_1)^2 \\ &-(a_2b_3c_3 + a_3b_2c_3 + a_3b_3c_2)^2(a_2b_1c_1 + a_1b_2c_1 + a_1b_1c_2)^2 \\ &-(a_3b_2c_2 + a_2b_3c_2 + a_2b_2c_3)^2(a_3b_1c_1 + a_1b_3c_1 + a_1b_1c_3)^2 \\ &+(a_2b_3c_3 + a_3b_2c_3 + a_3b_3c_2)(a_2b_1c_1 + a_1b_2c_1 + a_1b_1c_2) \\ &\qquad (a_3b_2c_2 + a_2b_3c_2 + a_2b_2c_3)(a_3b_1c_1 + a_1b_3c_1 + a_1b_1c_3) \\ &+(a_3b_2c_2 + a_2b_3c_2 + a_2b_2c_3)(a_3b_1c_1 + a_1b_3c_1 + a_1b_1c_3) \\ &\qquad (a_1b_3c_3 + a_3b_1c_3 + a_3b_3c_1)(a_1b_2c_2 + a_2b_1c_2 + a_2b_2c_1) \\ &+(a_1b_3c_3 + a_3b_1c_3 + a_3b_3c_1)(a_1b_2c_2 + a_2b_1c_2 + a_2b_2c_1) \\ &\qquad (a_2b_1c_1 + a_1b_2c_1 + a_1b_1c_2)(a_2b_3c_3 + a_3b_2c_3 + a_3b_3c_2). \end{aligned} \tag{5}$$

Expanding and collecting the above we get

$$S = 16\Big[\sum^{18} a_1^3a_2b_1b_2^2b_3c_2c_3^2 - \sum^{9} a_1^2a_2^2b_1b_2b_3^2c_1c_2c_3^2 - \sum^{9} a_1^4b_2^2b_3^2c_2^2c_3^2\Big], \tag{6}$$

or using the symbols of § 3,

$$S = 16(N - H - K). \tag{7}$$

* These two theorems lay no claim to novelty, see F. MORLEY, these Transactions, vol. 5 (1904), p. 472, for a discussion.

† *Höhere Ebene Kurven*, p. 247.

Employing the fact that the D's must enter in each term to an even power, if at all, we find that the only combinations entering are

$$I_2(D_1^2 + I_1^2) = 2N + 4E + 10F + 2G + 8H,$$
$$I_2^2 - D_2^2 = 4H,$$
$$(I_1^2 - D_1^2)^2 = 32N + 64H + 16K,$$
$$I_1 D_1 D_2 = -N + 2E + 5F + G - 4H.$$

It is not difficult to select the proper coefficients; the result is

$$S = 12I_2(D_1^2 + I_1^2) - 36(I_2^2 - D_2^2) - (I_1^2 - D_1^2)^2 - 24I_1 D_1 D_2 \tag{8}$$
$$= 4(I_1 D_1 - 3D_2)^2 - (I_1^2 + D_1^2 - 6I_2)^2. \tag{9}$$

Geometrically $S = 0$ is the condition that the Hessian of the Clebschian of the two 3-points shall be three lines.

Since the Clebschian treats the two triangles symmetrically, it is clear that S should be self dual. Applying the formulæ of § 3, we get

$$S' = D_1^4 S. \tag{10}$$

This is a good check on the work.

To substitute the coefficients of χ directly in SALMON's form for T* leads to complicated expressions. The following method presents fewer difficulties.

Let $(\alpha x)^3$ and $(a\xi)^3$ be two cubics giving the connex

$$(\alpha a)^2(\alpha x)(a\xi) = 0. \tag{11}$$

The 3 fixed points of the connex are given by

$$a_i(\alpha x)(\alpha a)^2 = \lambda x_i; \qquad (i = 1, 2, 3). \tag{12}$$

If the 3 equations of (12) are consistent

$$\Delta(\lambda) = \begin{vmatrix} a_1\alpha_1(\alpha a)^2 + \lambda & a_2\alpha_1(\alpha a)^2 & a_3\alpha_1(\alpha a)^2 \\ a_1\alpha_2(\alpha a)^2 & a_2\alpha_2(\alpha a)^2 + \lambda & a_3\alpha_2(\alpha a)^2 \\ a_1\alpha_3(\alpha a)^2 & a_2\alpha_3(\alpha a)^2 & a_3\alpha_3(\alpha a)^2 + \lambda \end{vmatrix} \equiv 0. \tag{13}$$

The coefficients of powers of λ in $\Delta(\lambda)$ are invariants of the connex. Now let $(\alpha x)^3$ become the reference 3-line and

$$\Delta(\lambda) = \begin{vmatrix} 2a_1a_2a_3 + \lambda & 2a_2^2a_3 & 2a_2a_3^2 \\ 2a_1^2a_3 & 2a_1a_2a_3 + \lambda & 2a_1a_3^2 \\ 2a_1^2a_2 & 2a_2^2a_1 & 2a_1a_2a_3 + \lambda \end{vmatrix}. \tag{14}$$

* *Höhere Ebene Kurven*, p. 248.

If the second cubic is taken as

$$(15)\qquad \begin{aligned} &a\xi_1^3 + b\xi_2^3 + c\xi_3^3 + 3a_2\xi_1^2\xi_2 + 3a_3\xi_1^2\xi_3 + 3b_1\xi_2^2\xi_1 \\ &\qquad + 3b_3\xi_2^2\xi_3 + 3c_1\xi_3^2\xi_1 + 3c_2\xi_3^2\xi_2 + 6m\xi_1\xi_2\xi_3 = 0, \end{aligned}$$

$\Delta(\lambda)$ takes the form, dropping numerical factors,

$$(16)\qquad \Delta(\lambda) = \begin{vmatrix} m+\lambda & a_3 & a_2 \\ b_3 & m+\lambda & b_1 \\ c_2 & c_1 & m+\lambda \end{vmatrix} = \lambda^3 + J_1\lambda^2 + J_2\lambda + J_3,$$

where

$$J_1 = 3m = 3I_1,$$

$$J_2 = 3m^2 - (b_1c_1 + c_2a_2 + a_3b_3),$$

$$J_3 = \begin{vmatrix} m & a_3 & a_2 \\ b_3 & m & b_1 \\ c_2 & c_1 & m \end{vmatrix}.$$

Now the Clebschian of the reference 3-line and the general cubic is

$$\chi = c_2x_1^2x_2 + b_1x_1x_3^2 + a_3x_2^2x_3 - c_1x_1x_2^2 - a_2x_3^2x_2 - b_3x_1^2x_3 = 0,$$

and T for this is

$$\begin{aligned} T = {} & 6b_1c_1c_2a_2a_3b_3 - 8(b_1^3c_1^3 + c_2^3a_2^3 + a_3^3b_3^3) - 27(a_3^2b_1^2c_2^2 + a_2^2b_3^2c_1^2) \\ & + 12(b_1^2c_1^2a_2c_2 + b_1^2c_1^2a_3b_3 + c_2^2a_2^2a_3b_3 + c_2^2a_2^2b_1c_1 + a_3^2b_3^2b_1c_1 + a_3^2b_3^2c_2a_2). \end{aligned}$$

Or we can write

$$(17)\qquad T = 6x - 8y - 27z + 12t.$$

From the expressions derived above we have

$$(18)\qquad (3I_1^2 - J_2)^3 = \sum^{3} b_1^3c_1^3 + 3\sum^{6} b_1^2c_1^2a_2c_2 + 6b_1c_1c_2a_2a_3b_3 = 6x + y + 3t,$$

$$(19)\qquad (J_3 + 2I_1^3 - I_1J_2)^2 = a_2^2b_3^2c_1^2 + a_3^2b_1^2c_2^2 + 2b_1c_1c_2a_2a_3b_3 = 2x + z,$$

$$(20)\qquad S(3I_1^2 - J_2) = -\sum^{3} b_1^3c_1^3 + 3b_1c_1c_2a_2a_3b_3 = 3x + y.$$

Elimination of x, y, z, t from these four equations gives

$$(21)\qquad T = 4(3I_1^2 - J_2)^3 - 27(J_3 + 2I_1^3 - I_1J_2)^2 + 12S(3I_1^2 - J_2).$$

But I_1 is identical with the I_1 of the present system, J_2 and J_3 are I_2 and $-I_3$ of HUN's system, and S is given by (9).

Substituting these values in (21) we find

$$
\begin{aligned}
T = &-8[(I_1^2 - D_1^2)^3 - 216 I_2^3 - 18 I_2 (I_1^4 - D_1^4) \\
&+ 108 (I_1^2 I_2^2 + D_1^2 D_2^2) - 54 (D_1 I_2 - I_1 D_2)^2 \\
&+ 36 I_1 D_1 D_2 (I_1^2 - D_1^2) - 648 I_2 D_2^2].
\end{aligned} \tag{22}
$$

Applying the dual formulas we have $T' = D_1^6 T$, again a check on the work.

As a further example, the invariant condition on the two triangles, that the two Clebschians may be apolar, will be calculated. The outline, only, of the work shall be given. The form of χ is given in full by equation (4). To get the equation of χ, the coefficients of the corresponding terms can be derived from χ by replacing each letter by its minor in D_1. Call the minors A_1, B_1, etc., as in § 3. Then

$$
\begin{aligned}
\chi' = &(A_2 B_3 C_3 + A_3 B_2 C_3 + A_3 B_3 C_2) \xi_1^2 \xi_2 + (A_1 B_2 C_2 + A_2 B_1 C_2 + A_2 B_2 C_1) \xi_1 \xi_3^2 \\
&+ (A_3 B_1 C_1 + A_1 B_3 C_1 + A_1 B_1 C_3) \xi_2^2 \xi_3 - (A_1 B_3 C_3 + A_3 B_1 C_3 + A_3 B_3 C_1) \xi_1 \xi_2^2 \\
&- (A_2 B_1 C_1 + A_1 B_2 C_1 + A_1 B_1 C_2) \xi_2 \xi_3^2 - (A_3 B_2 C_2 + A_2 B_3 C_2 + A_2 B_2 C_3) \xi_1^2 \xi_3.
\end{aligned}
$$

To find the apolarity condition, regard, as usual, the ξ's as differential operators and equate the result to zero. This gives

$$
\begin{aligned}
&(a_2 b_3 c_3 + a_3 b_2 c_3 + a_3 b_3 c_2)(A_2 B_3 C_3 + A_3 B_2 C_3 + A_3 B_3 C_2) \\
&+ (a_1 b_2 c_2 + a_2 b_1 c_2 + a_2 b_2 c_1)(A_1 B_2 C_2 + A_2 B_1 C_2 + A_2 B_2 C_1) \\
&+ (a_3 b_1 c_1 + a_1 b_3 c_1 + a_1 b_1 c_3)(A_3 B_1 C_1 + A_1 B_3 C_1 + A_1 B_1 C_3) \\
&+ (a_1 b_3 c_3 + a_3 b_1 c_3 + a_3 b_3 c_1)(A_1 B_3 C_3 + A_3 B_1 C_3 + A_3 B_3 C_1) \\
&+ (a_2 b_1 c_1 + a_1 b_2 c_1 + a_1 b_1 c_2)(A_2 B_1 C_1 + A_1 B_2 C_1 + A_1 B_1 C_2) \\
&+ (a_3 b_2 c_2 + a_3 b_3 c_2 + a_2 b_2 c_3)(A_3 B_2 C_2 + A_2 B_3 C_2 + A_2 B_2 C_3) = 0.
\end{aligned} \tag{23}
$$

By replacing the capitals by minors in the small letters and expanding, the whole can be summed thus:

$$
\begin{aligned}
&2(\sum^{9} a_1 a_2 a_3 b_2^2 b_3 c_1^2 c_3 - \sum^{6} a_1 a_2 a_3 b_1^2 b_3 c_2^2 c_3) + 3(\sum^{6} a_1 a_2^2 b_2 b_3^2 c_3 c_1^2 - \sum^{8} a_1 a_2^2 b_3 b_1^2 c_2 c_3^2) \\
&+ 2(\sum^{9} a_1^3 b_2 b_3^2 c_2^2 c_3 - \sum^{9} a_1^3 b_2^2 b_3 c_2 c_3^2) = 0.
\end{aligned} \tag{24}
$$

Expressed in terms of the fundamental invariants, this is

$$
D_1^3 - I_1^2 D_1 - 3 I_1 D_2 + 9 I_2 D_1 = 0. \tag{25}
$$

Hence: *The two Clebschians of two triangles are apolar if*

$$D_1^3 - I_1^2 D_1 - 3I_1 D_2 + 9I_2 D_1 = 0.$$

Of necessity, this form is self-dual.

The invariants of χ' are seen to be the dual of those of χ. But $S' = S$ and $T' = T$. Hence: *If neither triangle is degenerate, $S = 0$ is the condition that the Hessian of one Clebschian break down into 3 lines and at the same time that the Hessian of the other break down into 3 points.*

There are a number of other interesting cubic covariants and contravariants of two triangles. But the limits of this paper will not permit of a detailed treatment, the above discussion of the Clebschian being itself meager and incomplete.

§ 8. *Self-dual invariant forms.*

A number of self-dual forms have appeared in the course of this article, and the question naturally arises as to the number of such self-dual forms which are independent. Such forms are of special importance, because the vanishing of one of them indicates a mutual relation between the two triangles. The most general self-dual invariant of given degree can be found by carrying out the dual transformation on the most general invariant of that degree, and asking that the invariant be left unaltered save for multiplicative powers of D_1. For example, suppose the most general self-dual invariant of degree 3 is required. We ask that

$$\begin{aligned} &D_1^3(aI_1^3+bD_1^3+cI_1^2D_1+dI_1D_1^2+eI_1I_2+fI_1D_2+gD_1D_2+hI_2D_1+2kI_3) \\ &\quad\equiv a(I_1^3D_1^3-18I_1^2D_1^2D_2+108I_1D_1D_2^2-216D_2^3)+bD_1^6 \\ &\quad+cD_1^2(I_1^2D_1^2-12I_1D_1D_2+36D_2^2)+d(I_1D_1^5-6D_1^4D_2) \\ &\quad+e(I_1I_2D_1^3-2I_1^2D_1^2D_2+18I_1D_1D_2^2-6I_2D_1^2D_2-36D_2^3) \\ &\quad+f(-I_1D_1^3D_2+6D_1^2D_2^2)-g(D_1^4D_2)+h(D_1^4I_2-2I_1D_1^3D_2+6D_1^2D_2^2) \\ &\quad+k(2D_1^3I_3-2D_2^3+I_1D_1D_2^2-I_2D_1^2D_2). \end{aligned} \tag{1}$$

Equating coefficients, we get the equalities $c = -9a$, $k = 54a$, $6c = -(f+h)$, $g = -3d$. Introducing these in the left of (1), we have as the most general self-dual form of degree 3,

$$a(I_1^3-9I_1I_2+108I_3)+d(D_1^3-3D_1D_2)+c(I_1^2D_1-6I_1D_2)+h(I_2D_1-I_1D_2). \tag{2}$$

Since a, c, d and h are arbitrary constants it follows that the expressions in parentheses are all self-dual forms, as can be readily verified. In the further study of self-dual forms, we get the following system of five independent ones, which form a complete system:

$$(1)\ D_1, \qquad (2)\ I_1 D_1 - 3D_2, \qquad (3)\ I_1^2 - 6I_2,$$

$$(4)\ I_2 D_1 - I_1 D_2, \qquad (5)\ I_1^3 - 9I_1 I_2 + 108 I_3.$$

To prove this system complete let us designate forms (1), (2), (3), (4), (5) by A, B, C, E, and F respectively. We then have the following equations:

$$D_1 = A,$$

$$D_2 = -\tfrac{1}{3}(B - I_1 A) = \tfrac{1}{3}(AI_1 - B),$$

$$I_2 = \tfrac{1}{6}(I_1^2 - C),$$

$$I_3 = \tfrac{1}{108}(F - I_1 C + 3 I_1 I_2) = \tfrac{1}{216}(F - 3CI_1 + I_1^3).$$

That is, D_1, D_2, I_2, and I_3 can each be expressed in terms of I_1 and self-dual forms.

From (4) or E we get the further equation

$$E = \frac{A}{6}(I_1^2 - C) - \frac{I_1}{3}(AI_1 - B) \qquad \text{or} \qquad 6E = 2BI_1 - AC - AI_1^2,$$

and hence

$$(6) \qquad I_1^2 = \frac{6E + AC - 2BI_1}{A},$$

a formula which enables us to reduce higher powers of I_1. Hence from this series of equations we see that *any invariant can be expressed as a rational function of the five self-dual forms A, B, C, E, F, and of I_1 to the first degree.* That is any invariant I can be written as

$$(7) \qquad I = S_1 + kI_1 S_2,$$

where S_1 and S_2 are rational functions of A, B, C, E, and F. Now suppose I is self-dual and written in form (7). Since S_1 and S_2 have been proven self-dual, it follows that (7) can be true only for $kI_1 S_2 = 0$, as it can obviously not be self-dual, I_1 not being self-dual. Therefore if self-dual $I = S_1$, that is, a rational function of A, B, C, E, F. This proves the theorem.

Any form which can be expressed in terms of these five is self-dual, and it is frequently easier to show the self-duality of a form by expressing it in terms of

A, B, C, E, F, than by applying the dual transformation formulæ. Only the first two of the five have a simple geometrical interpretation known to the writer.

§ 9. *The fundamental invariants under a special Cremona transformation.*

It is the purpose here to show the effect of a special Cremona transformation on the fundamental invariants, and to apply the result to a single example.

Take the simple, quadratic involutory transformation

$$y_i = \frac{k}{x_i}.$$

The effect of this transformation on the invariants is to replace each symbol by its reciprocal. Since all invariant forms are homogeneous, k will appear to the same power in each term, hence may be taken as unity without loss of generality. The fundamental invariants, under this transformation, become

$$(1) \qquad \begin{aligned} D_1' &= \frac{D_2}{I_3}, \qquad I_1' = \frac{I_2}{I_3}, \\ D_2' &= \frac{D_1}{I_3}, \qquad I_2' = \frac{I_1}{I_3}, \qquad I_3' = \frac{1}{I_3}. \end{aligned}$$

As a typical illustration we take the rational quintic,

$$x_i = \frac{t - \alpha_i}{(t - \beta_i)^2} \qquad (i = 1, 2, 3).$$

This quintic has 3 cusps, 3 double points, 3 double tangents, 3 flexes, and is of class 5. It follows from the simple counting of constants that any two of the singular triangles can not be taken arbitrarily, but have a single relation connecting them. We shall give in detail but a single one of these relations. Taking the triangle of double points as the reference 3-line, we ask: What is the condition on the triangle of cusps?

From the theory of Cremona transformations,* it is well known that if the above involutory transformation is carried out on the given quintic, with the double points as fundamental points, the quintic goes into a rational quartic with 3 cusps, and with simple points at the 3 fundamental points. If we next carry out the same transformation regarding the cusps as fundamental points (which change of fundamental points is evidently equivalent to taking the dual), the rational quartic goes into a conic which *touches* the original fundamental lines, and has the three cusp points as ordinary points. But we already know

* CLEBSCH-LINDEMANN, vol. I, pp. 478, ff., where a number of references are given.

from an earlier paragraph the condition that a conic may be inscribed in the 3-line and circumscribed about the 3-point. It is

$$(2) \qquad (D_1^4 - 8D_1^2 I_2 + 16D_2^2)^2 - 512 I_3 D_1^3 (I_1 D_1 - 4D_2) = 0.$$

Hence to find the condition on the cusp triangle, we must take (2) and carry out the above transformations in reverse order. Being of period 2 the inverses are the same as the direct transformations. That is, transformation (1) must be carried out first, then the dual transformation, and finally (1) a second time. This gives an expression of the 24th degree in the coefficients of the original triangles.

If on the other hand the cusp triangle is taken as the reference 3-line the condition on the 3-point of double points turns out to be of the 18th degree by a similar argument. It is

$$D_1^8 D_2^5 + 16(I_1^2 D_1^2 D_2 - 4 I_1 D_1 D_2^2 + 4D_2^3 - D_1^4 D_2 - 16 I_3 D_1^3)$$

$$(3) \qquad \times (2D_1^3 I_3 - 2D_2^3 + I_1 D_1 D_2^2 - I_2 D_2^2 D_2)^2$$

$$-8D_2(2I_2 D_1^6 D_2 - 3I_1 D_1^5 D_2^2 + 6D_1^4 D_2^3)(2D_1^3 I_3 - 2D_2^3 + I_1 D_1 D_2^2 - I_2 D_1^2 D_2) = 0.$$

This is sufficient to illustrate our point. For by direct attack this problem is practically impossible, yet by this transformation it becomes comparatively simple. Other examples can readily be found to further illustrate this idea.

Incidentally this gives a convenient geometrical interpretation for $I_2 = 0$, the only one of the fundamental invariants not previously interpreted. For if the transformation $y_i = 1/x_i$ is carried out on the three points of the 3-point, they are carried into three new points, whose equations are

$$a_2 a_3 \xi_1 + a_3 a_1 \xi_2 + a_1 a_2 \xi_3 = 0,$$

$$b_2 b_3 \xi_1 + b_3 b_1 \xi_2 + b_1 b_2 \xi_3 = 0,$$

$$c_2 c_3 \xi_1 + c_3 c_1 \xi_2 + c_1 c_2 \xi_3 = 0.$$

Clearly the condition that these shall be apolar to the reference 3 line is

$$I_2 = 0.$$

Or as a definite theorem: *If two triangles are such that when one is taken as a 3-line and the other as a 3-point, and the transformation $y_i = 1/x_i$ is carried out, the 3-point is apolar to the 3-line, then will*

$$I_2 = 0.$$

This gives, however, no direct information as to the projective relation between the 3-line and 3-point in their original position. Other interpretations of $I_2 = 0$ arose, but none so direct and simple as for the vanishing of the other fundamental invariants.

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